SUP-NORM ESTIMATES FOR PARABOLIC SYSTEMS WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. We consider parabolic systems with nonlinear dynamic boundary conditions, for which we give a rigorous derivation. Then, we give them several physical interpretations which includes an interpretation for the porous-medium equation, and for certain reaction-diffusion systems that occur in mathematical biology and ecology. We devise several strategies which imply (uniform) L^p and L^∞ estimates on the solutions for the initial value problems considered.

1. Introduction

In this article, we consider the following system of quasilinear parabolic equations

$$\partial_t u_i - \Delta \left(A_i \left(u_i \right) \right) + f_i \left(x, t, \overrightarrow{u} \right) = 0, \text{ in } \Omega \times (0, \infty), \tag{1.1}$$

for $i = 1, ..., m$, where $\overrightarrow{u} = (u_1, ..., u_m)$, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with sufficiently smooth boundary $\Gamma := \partial\Omega$ (which is at least of class $\mathcal{C}^2$), for some given functions A_i and f_i. Denote by $\mathbb{N}_m = \{1, ..., m\}$ and consider two mutually disjoint (possibly empty) subsets $I_m, J_m \subseteq \mathbb{N}_m$ such that $I_m \cup J_m = \mathbb{N}_m$. Equation (1.1) is subject to the following set of boundary conditions

$$\partial_{\mathbf{n}} u_i + h_i \left(x, t, \overrightarrow{u} \right) = 0, \text{ on } \Gamma \times (0, \infty), \ \ i \in I_m \tag{1.2}$$

and

$$\delta_i \partial_t u_i + \partial_{\mathbf{n}} \left(A_i \left(u_i \right) \right) + g_i \left(x, t, \overrightarrow{u} \right) = 0, \text{ on } \Gamma \times (0, \infty), \ \ i \in J_m, \tag{1.3}$$

for some given functions g_i and h_i. Here $\delta_i > 0$ for $i \in J_m$, and we may assume, without loss of generality, that $\delta_i = 0$, for $i \in I_m$. The boundary conditions in (1.2)-(1.3) may be also mixed, that is, the boundary Γ may consists of two disjoint open subsets Γ_1 and Γ_2 on which the boundary conditions may be either of Dirichlet type or of the form (1.2) and (1.3), respectively. Finally, the model (1.1)-(1.3) could be also generalized be letting the reaction terms depend on advection, by allowing the diffusion rates depend also on x and t, or in other various ways. As usual, we equip the system (1.1)-(1.3) with the initial conditions

$$u_{i|t=0} = u_{i0} \text{ in } \Omega, \ u_{i|t=0} = v_{i0} \text{ on } \Gamma, \ i \in \mathbb{N}_m, \tag{1.4}$$

where in general, we may have $u_{i0|\Gamma} \neq v_{i0}$, $i \in \mathbb{N}_m$ (i.e., if u_{i0} is well-defined in the trace sense).

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We aim to give some results which allow to deduce L^{∞}-estimates for solutions of (1.1)-(1.4) assuming that some sort of energy estimate is apriori known in L^{p}-norm for some finite p. The main tool will be an iterative argument following a well-known Alikakos-Moser technique combined with a suitable form of Gronwall's inequality, which can then be used to prove that the L^{p}-L^{∞} smoothing property holds for any solutions of the *non-degenerate* parabolic system (1.1)-(1.4) (e.g., at least when $a_i(u_i) := A_i^{'}(u_i)$ satisfies (1.5) below). In order to deal with the full degenerate case (1.1)-(1.4) (at least in the case when $a_i(u_i) = |u_i|^{p_i}$, $p_i > 0$), we employ DeGiorgi's truncation method to prove the L^{p}-L^{∞} smoothing property. The precise statements of these results can be found in Section 3, see Theorems 3.1 and 3.2. A rigorous derivation and physical interpretation of the system (1.1)-(1.4) shall be given below in Section 2.

Why is it important to establish apriori (possibly, uniform in time) L^{∞}-estimates from some given L^{p}-estimate? To better give an idea of our larger scope let us take a look at some history for problems of the form (1.1)-(1.4). Problems such as (1.1)-(1.4) have already been investigated in a number of papers [16, 17, 8, 31] assuming that diffusion rates $a_i(u_i) := A_i^{'}(u_i)$ satisfy

$$a_i(u_i) \geq d_i > 0, \text{ for all } u_i \in \mathbb{R} \text{ and } i \in \mathbb{N}_m. \tag{1.5}$$

For instance, Constantin and Escher [16, 17] show that unique (classical) maximal solutions exist in some Bessel potential spaces under suitable assumptions on the nonlinearities f_i, g_i and h_i. Such results also enable the authors to investigate other qualitative properties concerning global existence and blow-up phenomena (see, also [8]). These results are also improved by Meyries [31], still in the non-degenerate case (1.5), by assuming more general boundary conditions (by also incorporating surface diffusion in (1.3)), and by requiring that the functions $f_i(\overrightarrow{u}), h_i(\overrightarrow{u}), g_i(\overrightarrow{u})$ are dissipative in a certain sense. However, none of these contributions deal with the degenerate case for equation (1.1), that is, when $a_i(u_i)$ is allowed to have a polynomial degeneracy at zero for some (if not all) $i \in \mathbb{N}_m$; for instance, one can take

$$a_i(u_i) = |u_i|^{p_i}, \ p_i > 0. \tag{1.6}$$

Moreover, it is well-known in the scalar case $m = 1$, that when at least one of the source terms, the bulk nonlinear term f_1 or the boundary term g_1 is present in (1.1)-(1.2), conditions can be derived on their growth rates which imply either the global existence of solutions or blow-up in finite time [21]. Namely in the non-degenerate case, for $\lambda, \mu \in \{0, \pm 1\}$ with $\max\{\lambda, \mu\} = 1$, $f_1(s) := -\lambda |s|^{r_1 - 1} s$ and $g_1(s) := -\mu |s|^{r_2 - 1} s$, solutions of

$$\partial_t u - \nu \Delta u + f_1(u) = h_1(x), \text{ in } \Omega \times (0, +\infty), \tag{1.7}$$

subject to the dynamic condition

$$\partial_t u + \nu b \partial_{\mathbf{n}} u + g_1(u) = h_2(x), \text{ on } \Gamma \times (0, \infty), \tag{1.8}$$

are globally well-defined, for every given (sufficiently smooth) initial data (1.4), if $r_1 r_2 > 1$ and $\lambda r_1 + \mu r_2 > 0$. Furthermore, [21] shows that if we further restrict the growths of r_1,r_2 so that $r_1 < (N+2)/(N-2)$ and $r_2 \leq N/(N-2)$, then the global solutions are also bounded. These restrictions can be eventually removed and more general conditions on f_1, g_1 can be deduced (see, e.g., [23]). On the other hand, if $\lambda = 0$, $\mu = 1$, then some solutions blowup in finite time with blowup occurring in the L^{∞}-norm at a rate $(t - T_*)^{-(r_2 - 1)}$, for some additional conditions

on u_0 and r_2. In the same way, when $\mu = 0$ and $\lambda = 1$, then some solutions blowup in finite time with a blowup rate which depends on r_1 and u_0 (see [3]). In the case when both $\mu \in \mathbb{R}$ and $\lambda > 0$ are nonzero, blowup may still occur for superlinear growth of f_1 and any growth of g (see [23]). The occurrence of blow up phenomena is closely related to the blowup problem for the ordinary differential equation

$$u_t + H(u) = 0,$$

where either $H = f_1$ or $H = g_1$. More precisely, it is easy to see that solutions of the ODE are spatially homogeneous solutions of either equation (1.7) or (1.8), and so if these solutions blowup in finite time so do the solutions of (1.7)-(1.8). It is worth mentioning that in [8] a criterion for the global existence of a (classical) maximal solution (on some interval $[0, t_+)$) to (1.7)-(1.8) is established using a variation of parameter formula. In particular, it is shown that if $t_+ < \infty$ then necessarily we must have

$$\limsup_{t \to t_+} \|u(t)\|_{L^\infty(\Omega)} = \infty.$$

Therefore, it appears that in order to deduce global existence of classical solutions to systems of the form (1.1)-(1.4), (1.5), it is generally required that we should deduce bounds on the solutions in L^∞-norm (see [31] also).

Finally, the L^p-L^∞ smoothing property also becomes an essential tool in attractor theory where it can be used to establish the existence of an absorbing set in L^∞-norm if this property can be deduced easily in L^p-norm for some finite p (in many applications in physics and mechanics, p is equal to either 1 or 2). Recall that a subset $\mathcal{B} \subset \mathcal{H}$, where $\mathcal{H}$ is a topological space endowed with a given metric, is called *absorbing* if the orbits corresponding to bounded sets $\mathcal{V}$ of initial data enter into $\mathcal{B}$ after a certain time (which may depend on the set $\mathcal{V}$) and will stay there forever. Moreover, we note that in order to study the long term behavior of the parabolic system (1.1)-(1.2), if the absorbing property holds in L^∞-norm, the growth rate of the nonlinearities f_i, g_i and h_i with respect to u_i becomes nonessential for further investigations of attractors. Indeed, the absorbing property can be also established in higher-order $W^{s,p}$-norms with relative ease provided that it is known in L^∞-norm. For the application of this property to attractor theory for parabolic equations of the form (1.7), (1.8), see [22, 23], where explicit dimension estimates for the global attractor for (1.7)-(1.8) are obtained.

The main goal of this paper is to deduce sufficiently general conditions on the diffusions and sources in (1.1)-(1.3), which would prevent blowup of any solution in the L^∞-norm, and show that the parabolic system under consideration is dissipative in a suitable sense. We outline the plan of the paper, as follows. In Section 2, we give the full derivation of systems of the form (1.1)-(1.3), and give physical interpretations to the dynamic boundary condition (1.2) for the porous-medium equation, and some models in ecology. In Section 3, after we introduce some notations and preliminary facts, we give the statements of our main results and some further applications. Finally, in Section 4 we provide the full proofs of these results.

2. Derivation and interpretation

Let $\Gamma \subset \mathbb{R}^{N-1}$ consists of two disjoint open subsets Γ_1 and Γ_2, each $\overline{\Gamma}_i \backslash \Gamma_i$ is a S-null subset of Γ and $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ with $\Gamma_1 \subseteq \Gamma$. We shall only give the derivation in the case of the scalar equation

$$\partial_t u - \operatorname{div}(a(u) \nabla u) + f(u) = h_1(x), \tag{2.1}$$

equipped with (nonlinear) dynamic boundary conditions

$$\partial_t u + a(u) \nabla u \cdot \mathbf{n} + g(u) = h_2(x), \tag{2.2}$$

on Γ_1, and Dirichlet boundary conditions

$$u_{|\Gamma_2} = 0. \tag{2.3}$$

We can easily extend our arguments to systems as well (see below). Equations (2.1)-(2.3) are also subject to the initial conditions

$$u_{|t=0} = u_0 \text{ in } \Omega, \quad u_{|t=0} = v_0 \text{ on } \Gamma. \tag{2.4}$$

Standard derivations of the porous medium equation always use the principle "amount of fluid in equals amount of fluid out" over a region Ω, occupied by either a liquid or gas, and is based on the fact that this fluid diffuses from locations of higher to those of lower pressure. In the traditional approach, the porous medium equation is assumed to hold in the region Ω and then the boundary conditions are appended later. There are three standard boundary conditions that specify the density on the boundary of Ω; they are Dirichlet and Neumann-Robin type of boundary conditions (see, e.g., [38]). Dynamic boundary conditions for porous medium equations seem to have appeared before in different contexts [6, 19, 36, 35]. For instance, [6] deals with the modelling of the rain water infiltration through the soil above an aquifer in regimes where there is runoff at the ground surface. In general, all rain water infiltrates into the soil, but if the rainfall event is particularly intense, the maximum draining capacity of the soil is exceeded. In this case, dynamic boundary conditions (see (2.8) below) are needed to describe the saturation of layers near the ground surface (cf. also [19, 36]). Porous-medium like systems (2.1)-(2.3) can also be found as part of (larger) coupled systems of partial differential equations (such as, (1.1)-(1.4)) describing the vertical movement of water and salt in a domain splitted in two parts: a water reservoir and a saturated porous medium below it, in which a continuous extraction of fresh water takes place (for instance, by the roots of mangroves) [24]. Such problems are formulated in terms of equations for the salt concentration and the water flow in the porous medium, with a dynamic boundary condition which connects both subdomains. Finally, dynamic boundary conditions similar to (2.2) also appear in certain classes of parabolic equations with boundary hysteresis (see, e.g., [35, Section 4] and the references therein). For some applications of dynamic boundary conditions for physiologically structured populations with diffusion we refer the reader, for instance, to [20].

For all the phenomena of the kind discussed here, the method of introducing dynamic boundary conditions seems ad hoc. It would be more natural if such boundary conditions could be derived in the context of energy balance and constitutive laws. Moreover, the usual derivation of the porous medium equation with standard boundary conditions does not show how to model, for instance, a water source, which is located on the boundary of Ω. To this end, we shall rethink the usual derivation of the porous medium equation (2.1), by making essential connections between the differential equation (2.1) and the boundary conditions (2.2)-(2.3), and, thus, try to convince the reader that our new perspective is more natural than the traditional way. Let $p(x,t)$ denote the pressure of fluid at $x \in \Omega$ and time $t > 0$. Consider the mass of fluid in an element of volume V given by

$$\int_V \alpha(x) u(x,t) \, dx,$$

where $\alpha(x) > 0$ defines the porosity of medium at the point $x \in \Omega$. Similarly, we define

$$\int_{\Gamma} \beta(x)\, v(x,t)\, dS$$

as the mass of fluid across the surface Γ, where $\beta(x)$ is such that

$$\Gamma_3 := \{x \in \Gamma_1 : \beta(x) > 0\}$$

is a set of positive measure and $\Gamma_3 \subseteq \Gamma_1$. In what follows, we shall take $\Gamma_2 = \varnothing$ for the sake of exposition, so that $\Gamma_1 \equiv \Gamma$. The flux $\mathbb{J}(x,t)$, at which the fluid moves across a surface element S with normal $\mathbf{n}$, is given by

$$\int_S \mathbb{J}(x,t) \cdot \mathbf{n} dS.$$

Suppose now there is a source on the boundary Γ to be represented by a function

$$\Psi = \Psi(t, x, u, \nabla u).$$

The amount of fluid leaving the region is still given by $\int_\Gamma \mathbb{J}(x,t) \cdot \mathbf{n} dS$, but the amount of fluid leaving into the region must also take into account the action of the source Ψ on Γ. It is worth pointing out that, in practice, when rainfall only partially infiltrates the soil, the water will accumulate on the ground surface Γ_1 as the surface layer becomes saturated; hence, necessarily, $\Psi \neq 0$ on Γ. We use the measure space $(\overline{\Omega}, d\mu)$ which we redefine as $(\Omega, dx) \oplus (\Gamma, dS)$. Then the conservation of fluid in $\overline{\Omega}$ takes the form

$$\begin{aligned} \partial_t \left(\int_\Omega \alpha(x)\, \rho dx + \int_\Gamma \beta(x)\, \eta dS \right) + \int_\Gamma \mathbb{J} \cdot \mathbf{n} dS \\ = \int_\Omega \Xi dx + \int_\Gamma \Psi dS, \end{aligned} \tag{2.5}$$

where $\Xi = \Xi(x,t,u)$ denotes any volume source density function. Notice that equation (2.5) must also account for a term like

$$\int_\Gamma \beta(x)\, v dS,$$

due to the presence of the source density Ψ at Γ. Assuming that $\mathbb{J}$ is sufficiently smooth and applying the divergence theorem in (2.5), we deduce

$$\begin{aligned} \partial_t \left(\int_\Omega \alpha(x)\, u dx + \int_\Gamma \beta(x)\, u dS \right) + \int_\Omega div(\mathbb{J})\, dx \\ = \int_\Omega \Xi dx + \int_\Gamma \Psi dS. \end{aligned} \tag{2.6}$$

Assuming that the density functions u, v are also differentiable with respect to $t > 0$ and since (2.6) holds for any subdomain $\Omega_0 \subseteq \Omega$, the usual argument yields the following differential equation

$$\partial_t (\alpha(x)\, u(x,t)) = -\mathrm{div}(\mathbb{J}(x,t)) + \Xi(x,t,u(x,t)),\ x \in \Omega,\ t > 0. \tag{2.7}$$

Then, from (2.6) the boundary condition becomes

$$\int_\Gamma \left[\partial_t (\beta(x)\, u(x,t)) - \Psi(x,t,u(x,t), \nabla u(x,t))\right] dS = 0,\ t > 0,$$

which clearly holds if

$$\partial_t (\beta(x)\, u(x,t)) - \Psi(x,t,u(x,t), \nabla u(x,t)) = 0, \text{ for } x \in \Gamma,\ t > 0. \tag{2.8}$$

Darcy's law states that the flux $\mathbb{J}$ depends on the pressure gradient so it takes the form

$$\mathbb{J}(x,t) = -\frac{\mathcal{K}(x)}{\nu} u(x,t) \nabla p(x,t), \ x \in \Omega, \ t > 0, \tag{2.9}$$

where $\nu > 0$ is the viscosity of the fluid and $\mathcal{K}(x)$ defines the permeability of the porous medium. Finally, if one also makes the assumption that the pressure p is described by an equation of state involving the density, $p = b(u)$, then substituting the appropriate quantities in (2.7), we obtain the following porous medium equation (2.10)

$$\partial_t (\alpha(x) u(x,t)) = \frac{1}{\nu} \mathrm{div}\left(\mathcal{K}(x) a(u(x,t)) \nabla u(x,t)\right) + \Xi(x,t,u(x,t)), \text{ in } \Omega, \ t > 0,$$

where we have set $a(t) \equiv tb(t)$. The function b that relates the density to pressure is, in general, monotone and, in fact, strictly increasing in many applications of the type of fluid being considered in the literature.

Finally, let us now focus on the boundary condition (2.8). We now show that a quite large class of boundary conditions for equation (2.1) can be written in this way for various choices of Ψ. We emphasize that in this formulation the boundary conditions arise naturally in the formulation of the problem. Suppose first $\Gamma_3 \equiv \Gamma_1$ (i.e., $\beta(x) > 0$ a.e. in Γ_1) and $\beta \in C^1(\Gamma_1)$. Choosing $\Psi \equiv 0$ (i.e., no source is located at Γ_1), so that $\partial_t u \equiv 0$ on $\Gamma_1 = \Gamma$, therefore

$$u(x,t) = u_0(x), \tag{2.11}$$

for $x \in \Gamma_1$ and $t \geq 0$, where u_0 is the initial condition associated with equation (2.1). Thus, we obtain a Dirichlet boundary condition for u. In order to derive an inhomogeneous Neumann boundary condition, we suppose that Ψ only depends on t. Then, if u is sufficiently regular, we have $\partial_t u(x,t) = (1/\beta(x)) \Psi(t)$ on Γ_1, for any $t > 0$. Hence, if Γ_1 is smooth enough as well, we have $\partial_t (\nabla u) = \nabla(\partial_t u) = \boldsymbol{\gamma}(x) \Psi(t)$ on $\overline{\Omega} \times (0,+\infty)$, for some $\boldsymbol{\gamma}(x) \in \mathbb{R}^N$. This entails that $\nabla u(x,t) = \mathbf{H}(x,t)$ holds for $(x,t) \in \Gamma_1 \times (0,+\infty)$ and some smooth function $\mathbf{H} \in \mathbb{R}^N$. Therefore, we have

$$\nabla u(x,t) \cdot \mathbf{n} = \mathbf{H}(x,t) \cdot \mathbf{n}, \quad (x,t) \in \Gamma_1 \times (0,+\infty). \tag{2.12}$$

To obtain a Robin boundary condition, we set $\Psi(x,t,u,\nabla u) = \beta(x) e^{Cr} q(t)$, for some $C \in \mathbb{R}$, where r is defined as the parameter describing the line ℓ which passes through x and contains $\mathbf{n}$ such that $r > 0$ at all points on $\ell \cap \Omega$ which are close to x. We thus have $\partial_t u(x,t) = e^{Cr} q(t)$ which implies

$$\nabla(\partial_t u(x,t)) \cdot \mathbf{n}(x,t) = \partial_t (\nabla u(x,t) \cdot \mathbf{n})(x,t) = Ce^{Cr} q(t),$$

for $(x,t) \in \Gamma_1 \times (0,+\infty)$. Therefore, we infer

$$\partial_t (\nabla u(x,t) \cdot \mathbf{n})(x,t) - C \partial_t u(x,t) = 0, \qquad \text{on } \Gamma_1 \times (0,+\infty),$$

so that

$$\nabla u(x,t) \cdot \mathbf{n} - Cu(x,t) = j(x), \quad (x,t) \in \Gamma_1 \times (0,+\infty). \tag{2.13}$$

We have thus recovered the most common boundary conditions. On the other hand, in order to model a water source placed on the boundary Γ_1, we may assume that Ψ only depends nonlinearly on the flux $\nabla u \cdot \mathbf{n}$ across the boundary as well as on a nonlinear source $g(u)$, which may represent effects of reaction or absorption at Γ_1. That is, we let

$$\Psi(x,u,\nabla u) = -a(u) \nabla u \cdot \mathbf{n} - g(u) - h_2(x).$$

Then, the resulting boundary condition for equation (2.1) becomes

$$\beta(x)\,\partial_t u(x,t) + a(u(x,t))\,\nabla u(x,t)\cdot\mathbf{n} + g(u(x,t)) = h_2(x)\,, \tag{2.14}$$

for $(x,t)\in\Gamma_1\times(0,+\infty)$. This is a reasonably general condition which contains the usual (homogeneous) ones along with the so-called dynamic boundary condition when $\Gamma_3=\Gamma_1\equiv\Gamma$. When ponding or surface runoff occurs at the surface Γ_1, it also includes the dynamic boundary condition contained in Filo-Luckhaus [19]. Finally, we notice that $\beta(x)$ may be such that $\Gamma_3\neq\Gamma_1$, so that we can also cover the case where the boundary conditions (2.14) are dynamic only on a part of the boundary of Γ_1. If the source Ψ also depends on v in the tangential coordinates of the boundary, i.e., $\Psi=\Psi(x,u,\nabla u,\nabla_{\Gamma_1}u)$, then we can also model local diffusion in (2.14) by incorporating the elliptic Laplace-Beltrami operator Δ_{Γ_1} (or other nonlinear differential operators) on the manifold $\Gamma_1\subset\mathbb{R}^{N-1}$.

We will now give a physical interpretation of the effect of a water source on the patch Γ_1, at least in some cases. A similar approach was also used in the derivation of heat and wave equations (see, e.g., [25]). We will mainly focus on the following boundary condition

$$\partial_t u + a(u)\,\nabla u\cdot\mathbf{n} = 0, \text{ on } \Gamma_1\times(0,\infty)\,. \tag{2.15}$$

We work in an infinitesimal region on the boundary. Choose a point $x\in\Gamma_1$ and let $B_\kappa(x)$ be a ball of radius $\kappa>0$ about x. Since Γ_1 is regular, we can choose a coordinate system for $B_\kappa(x)\cap\overline{\Omega}$ so that the boundary of $B_\kappa(x)\cap\overline{\Omega}$ in the transformed coordinate system is flat, x is mapped to $\overline{x}=(x_1,x_2,...,x_{N-1},0)$, that is, the boundary Γ, at least locally near x lies on the hyperplane $x_N=0$. Then the outward unit normal $\mathbf{n}$ to Γ at x is the unit vector in the direction of e_N which we will denote by r. Then, locally near x, (2.15) becomes

$$\partial_t u + \partial_r(A(u)) = 0, \;\; (r,t)\in(0,r_0)\times(0,t_0)\,, \tag{2.16}$$

for some sufficiently small positive constants r_0, t_0. Observe that (2.16) resembles nothing more than a scalar *conservation law*, where we have set $A(u)=\int_0^u a(t)\,dt$. Equation (2.16), subject to initial condition $u(r,0)=v_0(r)$, $r\in(0,r_0)$, possesses interesting types of fundamental solutions such as, travelling waves that describe the movement of a mass of fluid in the direction of the unit normal $\mathbf{n}\in\mathbb{R}^N$, and source-type solutions starting from a finite mass concentrated at a single point of space, say, $v_0(r)=\mathcal{C}\delta_0(r)$, $\mathcal{C}>0$.

In the latter case, explicit self-similar solutions of the form $u(r,t)=\Theta(r/t)$ are well-known to exist, as a consequence of the invariance of (2.16) under the scaling $(r,t)\longmapsto(\lambda r,\lambda t)$, $\lambda\in\mathbb{R}$. More precisely, from (2.16) Θ clearly has to satisfy

$$\zeta\Theta'(\zeta) - (A(\Theta(\zeta)))' = 0$$

and this yields, formally, to

$$u(r,t)=\Theta(r/t)=(A')^{-1}(r/t)\,,$$

as long as, $(A')^{-1}$ is well defined at least in a sufficiently small real interval. In the former case, one can search for particular solutions of (2.16) in the form $u(r,t)=\eta(r-ct)$, where $c\in\mathbb{R}$ is the speed of the travelling wave and η has to be determined. Substituting the expression for $u(r,t)$ in (2.16), we deduce that c is an eigenvalue (with η' as eigenvector) for

$$(-c+a(\eta(r-ct)))\,\eta'(r-ct)=0.$$

Under appropriate structural conditions on $a(\cdot)$ (see, e.g., Section 3), this eigenvalue problem is strictly hyperbolic (cf., e.g., [18, Chapter 11]) with speed $c = c(\eta) > 0$, hence a wave-like solution $\eta = \eta(r - ct)$ to (2.16) can be found. This is a unidirectional wave which travels *into* the region Ω. We can now map back to our original coordinate system to find that $u(r,t) = \eta(x - ct\mathbf{n})$ is a solution to (2.15). In plain physical terms, the mass of fluid is carried by the wave η into an infinitesimal layer near the boundary Γ. This wave will cease to exist after some small time since once inside Ω, the primary process is governed by nonlinear diffusion in the porous medium equation (2.10).

It is easy to extend our derivation to systems of the form (1.1)-(1.4) and to give them phyiscal interpretations. For instance, these systems also occur in the pharmaceutical industry by mathematical modells for the development of blood coagulation treatments with specific coagulation factors [14, 28, 32]. The systems (1.1)-(1.4) are also motivated by diffusion processes on metric graphs and ramified spaces, which yield interface problems for quantum graphs with coupled dynamic boundary conditions at the nodes (see, e.g., [33] and references therein). On the other hand, the reaction-diffusion equations (1.1) (for $A_i(u_i) = d_i u_i$, $i \in \mathbb{N}_m$) arise as models for the densities u_i, $i \in \mathbb{N}_m$ of substances or organisms that disperse through space by Brownian motion, random walks, hydronamic turbulance or similar mechanisms. These equations are widely used as models to account for spatial effects in ecological enviroments [9]. For equations (1.1), the Dirichlet boundary condition (2.11) specifies the density u_i of species at the boundary Γ, with an interpretation that anything that reaches the boundary Γ of Ω leaves and does not return. If $u_0 \equiv 0$, then (2.11) may be interpreted as if the species suffers extiction if say the patch Γ_1 where the individuals live is toxic. The (homogeneous, $\mathbf{H} \equiv \mathbf{0}$) Neumann boundary condition (2.12) says that nothing can cross the boundary of Ω. Another relation in ecological models is the Robin boundary condition (2.13) with $j(x) \equiv 0$ and $C = C_i \in \mathbb{R}^*$, $i \in \mathbb{N}_m$ which can be interpreted as saying that when organisms reach the boundary some leave it but some do *not* depending on the sign of C_i. Finally, the other *not* so common condition is the Wentzell-type (dynamic) boundary condition (2.14) which states that change in the density of individuals at Γ_1 is a function of their flux in the normal direction across Γ_1 and some other function of density if no dispersive effects along Γ_1 are taken into account. Following our reasonning above, this type of boundary condition (1.3) can be interpreted as saying that some individuals may choose to live on the patch Γ_1 but some may *not* and can choose to return to the region Ω, where spatial diffusion coupled with reaction in the bulk Ω is the main mechanism for population movement and interaction. Suppose, for instance, that certain critical resources for a specific population u_i, $i \in \mathbb{N}_m$, are available only on Γ_1. Then u_i must obey the rule

$$\partial_t u_i + d_i \nabla u_i \cdot \mathbf{n} + h(x) u_i = 0, \text{ on } \Gamma_1 \times (0, \infty), \tag{2.17}$$

which says that the density u_i diffuses (in an infinitesimal layer near Γ_1) toward the patch Γ_1 in the direction of normal flux. Again the main mechanism for this behavior here is the influence of external forces on Γ on a particular population u_i. Of course, in this context the function $h(x)$ plays the role of a resource density function on Γ_1, and it can generally depend also on time. In fact, it is not hard to imagine a typical scenario where predatory individuals are preferentially concentrated around valued resources on Γ_1 where the likelihood of prey is greatest. Hence, in the more general case of (2.14) the state densities u_i may be also allowed to carry mass on

Γ_1 in contrast to the usual Robin condition for which the mass is always zero. This general description (1.3) along the patch Γ_1 can have substantial consequences on the dynamics of various ecological enviroments modelled by reaction-diffusion systems. We give a short reasonning for this behavior as follows. In the case of a scalar (non-degenerate) diffusion equation ($m = 1$, $a_1(\cdot) \equiv d_1$), we have shown in [22, 23] (say, in dimension $N \geq 3$) that problem (1.7)-(1.8) posseses a finite dimensional global attractor $\mathcal{A}_{\rm dyn}$ whose dimension is essentially of *different order* than the dimension of the global attractor $\mathcal{A}_{\text{D-N-R}}$ for the same parabolic problem (1.7) with a Dirichlet/Neumann-Robin boundary condition (2.11)-(2.13) (cf., also [23]). In particular, the correct asymptotics for the Hausdorff and, respectively, the fractal dimensions of $\mathcal{A}_{\rm dyn}$ are

$$\dim_H \mathcal{A}_{\rm dyn} \sim C(f_1, g_1) \frac{|\Gamma|}{(\nu b)^{N-1}}, \ \dim_F \mathcal{A}_{\rm dyn} \sim C(f_1, g_1) \frac{|\Gamma|}{(\nu b)^{N-1}}, \tag{2.18}$$

as long as $\nu \to 0^+$. Here, $C = C(f_1, g_1)$ is a positive constant that is independent of the size of Ω, but depends only on f_1 and g_1, and $|\Gamma|$ denotes the natural Lebesgue surface measure of $\Gamma \subset \mathbb{R}^2$. Note that the asymptotics for the dimension of $\mathcal{A}_{\text{D-N-R}}$ is actually $C(f_1) |\Omega| / (\nu b)^{N/2}$, as $\nu \to 0^+$ (see, e.g., [5]) suggesting that the dynamics on $\mathcal{A}_{\rm dyn}$ is *qualitatively* different than that on $\mathcal{A}_{\text{D-N-R}}$ even though both systems are gradient like [23] (i.e., both problems possess a global Lyapunov function). The asymptotic estimates in (2.18) are essentially determined by the instability indices of a properly chosen family of (hyperbolic) equilibria u_* (see, e.g., [5], [22]). One achieves a lower bound like (2.18) by computing the dimension of the unstable eigenspace E^u of the linearization of (1.7)-(1.8) around a constant equilibrium u_*. In this case, the linearized system possesses at least $n \sim C(f_1, g_1) |\Gamma| / (\nu b)^{N-1}$ (as $\nu \to 0^+$) unstable solutions. This points out once again to the destabilizing nature of the dynamic boundary condition (2.17) even when the dynamics in the bulk Ω is essentially strictly *linear* (see, Appendix). We emphasize that this kind of behavior *cannot* hold for the Dirichlet/Neumann-Robin boundary condition (2.11)-(2.13).

3. Main results

The natural phase-space for problems of the form (1.1)-(1.4) is

$$\mathbb{X}^{s_1,s_2} := L^{s_1}(\Omega) \oplus L^{s_2}(\Gamma) = \{U = \binom{u_1}{u_2} : \ u_1 \in L^{s_1}(\Omega), \ u_2 \in L^{s_2}(\Gamma)\},$$

$s_1, s_2 \in [1, +\infty]$, endowed with norm

$$\|U\|_{\mathbb{X}^{s_1,s_2}} = \left(\int_\Omega |u_1(x)|^{s_1} dx\right)^{1/s_1} + \left(\int_\Gamma |u_2(x)|^{s_2} dS_x\right)^{1/s_2}, \tag{3.1}$$

if $s_1, s_2 \in [1, \infty)$, and

$$\begin{aligned} \|U\|_{\mathbb{X}^\infty} &:= \max\{\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Gamma)}\} \\ &\simeq \|u_1\|_{L^\infty(\Omega)} + \|u_2\|_{L^\infty(\Gamma)}. \end{aligned}$$

We agree to denote by $\mathbb{X}^s$ the space $\mathbb{X}^{s,s}$. Moreover, we have

$$\mathbb{X}^s = L^s\left(\overline{\Omega}, d\mu\right), \ s \in [1, +\infty], \tag{3.2}$$

where the measure $d\mu = dx_{|\Omega} \oplus dS_{x|\Gamma}$ on $\overline{\Omega}$ is defined for any measurable set $A \subset \overline{\Omega}$ by

$$\mu(A) = |A \cap \Omega| + S(A \cap \Gamma). \tag{3.3}$$

Identifying each function $\theta \in C\left(\overline{\Omega}\right)$ with the vector $\Theta = \binom{\theta_{|\Omega}}{\theta_{|\Gamma}}$, we have that $C(\overline{\Omega})$ is a dense subspace of $\mathbb{X}^s$ for every $s \in [1, \infty)$ and a closed subspace of $\mathbb{X}^\infty$. In general, any vector $\theta \in \mathbb{X}^s$ will be of the form $\binom{\theta_1}{\theta_2}$ with $\theta_1 \in L^s(\Omega, dx)$ and $\theta_2 \in L^s(\Gamma, dS)$, and there need not be any connection between θ_1 and θ_2.

Next, we set

$$\mathcal{X}^{s_1,s_2} := \prod_{i \in I_m} L^{s_1}(\Omega) \times \prod_{i \in J_m} \mathbb{X}^{s_1,s_2}, \text{ for any } s_1, s_2 \in [1, +\infty],$$

where, for any given set X,

$$\prod_{i \in I} X := \underbrace{X \times ... \times X}_{|I| \text{ times}}.$$

The norm in the space $\mathcal{X}^{s_1,s_2}$, for any $s_1, s_2 \in [1, +\infty)$ is

$$\|\overrightarrow{u}\|_{\mathcal{X}^{s_1,s_2}} := \sum_{i \in I_m} \|u_i\|_{L^{s_1}(\Omega)} + \sum_{i \in J_m} \left(\|u_i\|_{L^{s_1}(\Omega)} + \delta_i \|u_i\|_{L^{s_2}(\Gamma)} \right), \tag{3.4}$$

where $\overrightarrow{u} = (u_1, ..., u_m)$, while the norm in the space $\mathcal{X}^\infty := \mathcal{X}^{\infty,\infty}$ is naturally given by

$$\|\overrightarrow{u}\|_{\mathcal{X}^\infty} := \max\{\max_{i \in \mathbb{N}_m} \|u_i\|_{L^\infty(\Omega)}, \max_{i \in J_m} \|u_i\|_{L^\infty(\Gamma)}\}.$$

If $s_1 = s_2 = s$, we will simply write $\mathcal{X}^s$ instead of $\mathcal{X}^{s_1,s_2}$. Finally, without further abuse of notation, we will also refer to $\mathcal{X}^{\overrightarrow{r}}$, $\overrightarrow{r} = (r_1, ..., r_m)$, as the following Banach space

$$\mathcal{X}^{\overrightarrow{r}} := \prod_{i \in I_m} L^{r_i}(\Omega) \times \prod_{i \in J_m} \mathbb{X}^{r_i,r_i},$$

endowed with the natural norm in (3.4).

Let us now state our main hypotheses on the source terms f_i, g_i, h_i and nonlinear diffusions a_i, for each $i \in \mathbb{N}_m$.

Conditions on a_i : The Carathéodory functions a_i (with values in $\mathbb{R}$) satisfy the condition: $\exists \alpha_i > 0$, $\forall s_i \in \mathbb{R}$ such that

$$a_i(s_i) \geq \alpha_i |s_i|^{p_i}, \ i \in \mathbb{N}_m, \tag{3.5}$$

for some nonnegative p_i.

Conditions on f_i, g_i, h_i : The Carathéodory functions f_i, g_i, h_i (with values in $\mathbb{R}$) satisfy the conditions: $\exists C_{f_i}, C_{g_i}, C_{h_i} > 0$, for almost all (x,t), $\forall s_i \in \mathbb{R}$, such that

$$\begin{cases} \sum_{i \in \mathbb{N}_m} f_i(x,t,s_1,...,s_m) s_i \geq -\sum_{i \in \mathbb{N}_m} C_{f_i} |s_i|^2 - \widetilde{C}_f, \\ \sum_{i \in J_m} g_i(x,t,s_1,...,s_m) s_i \geq -\sum_{i \in J_m} C_{g_i} |s_i|^2 - \widetilde{C}_g, \\ \sum_{i \in I_m} h_i(x,t,s_1,...,s_m) s_i \geq -\sum_{i \in I_m} C_{h_i} |s_i|^2 - \widetilde{C}_h, \end{cases} \tag{3.6}$$

for some nonnegative $\widetilde{C}_f$, $\widetilde{C}_g$ and $\widetilde{C}_h$.

The question of global existence and the L^p-L^∞ smoothing property for solutions of (1.1)-(1.4) can be stated for the function $\overrightarrow{u} = (u_1, ..., u_m)$, as follows. We say that the parabolic system (1.1)-(1.4) satisfies **Property P**(r_1, r_2), for some finite $r_1, r_2 \geq 1$, if, for all $i \in \mathbb{N}_m$, any of the following conditions holds:

(i) There exists a positive function Q, independent of initial data, and a positive constant η, such that

$$\sup_{t \geq \eta > 0} \|\overrightarrow{u}(t)\|_{\mathcal{X}^{r_1,r_2}} \leq Q(\eta). \tag{3.7}$$

(ii) There exists a positive constant $\mathcal{C}$, independent of initial data, such that

$$\limsup_{t \to \infty} \|\overrightarrow{u}(t)\|_{\mathcal{X}^{r_1,r_2}} \leq \mathcal{C}. \tag{3.8}$$

(iii) If $\overrightarrow{u}_0 = (u_{10}, ..., u_{m0}) \in \mathcal{X}^\infty$, $i \in \mathbb{N}_m$, there exists a positive function Q, independent of initial data, such that

$$\sup_{t\geq 0} \|\overrightarrow{u}(t)\|_{\mathcal{X}^{r_1,r_2}} \leq Q\left(\|\overrightarrow{u}_0\|_{\mathcal{X}^\infty}\right). \tag{3.9}$$

The first result concerning the L^∞-estimate for solutions of (1.1)-(1.4) in the non-degenerate case, shows that if either one of the properties for $\mathbf{P}(s_1, s_2)$ above holds apriori for some finite $s_1, s_2 \geq 1$, then it also holds for $s_1 = s_2 = \infty$.

Theorem 3.1. *Let the assumptions (3.5), (3.6) be satisfied such that $p_i = 0$, for all $i \in \mathbb{N}_m$ (i.e., (1.5) holds). Suppose that the system (1.1)-(1.4) satisfies the property $\boldsymbol{P}(1,1)$-(i) (respectively, (ii) or (iii)), then it also satisfies $\boldsymbol{P}(\infty,\infty)$-(i) (respectively, (ii) or (iii)).*

Remark 3.1. Note that we can also consider the more general case in which $a_i(u_i)$ is replaced by $a_i(x, t, \overrightarrow{u})$, $i \in \mathbb{N}_m$, that is, the equations (1.1) are strongly coupled in their diffusions. In this case, we must replace assumption (3.5) by

$$a_i(x, t, \overrightarrow{s}) \geq \alpha_i |\overrightarrow{s}|^{p_i}, \ \forall \overrightarrow{s} \in \mathbb{R}^m, \tag{3.10}$$

for almost all (x, t), and notice that all the computations performed in the proof of Theorem 3.1 hold automatically since $|\overrightarrow{s}|^{p_i} \geq |s_i|^{p_i}$ for any $\overrightarrow{s} = (s_1, ..., s_m) \in \mathbb{R}^m$ (of course, in that case $p_i = 0$ by assumption, for all $i \in \mathbb{N}_m$). Theorem 3.1 can be also extended to systems with p-Laplacian diffusions, i.e.,

$$a_i = a_i\left(u_i, |\nabla u_i|^{\varrho_i - 2}\right) \geq \alpha_i |\nabla u_i|^{\varrho_i - 2},$$

for some $\varrho_i \geq 2$, $i \in \mathbb{N}_m$, by following, for instance, [23].

The second result is concerned with the full degenerate case (1.1)-(1.4) when the assumptions of Theorem 3.1 do not hold (in particular, if it happens that $p_i > 0$ for some $i \in \mathbb{N}_m$). The proof is based on a truncation technique which was originally developed by DeGiorgi to study the regularity of solutions to elliptic equations, and then extensively used by many authors to study weak solutions to degenerate parabolic systems, subject to the usual static boundary conditions (see, e.g., [10, 27]). Here, we extend DeGiorgi's method to problems of the form (1.1)-(1.4). In order to avoid additional technicalities due to the different conditions that one can assign on the boundary Γ for each u_i, $i \in \mathbb{N}_m$, we shall focus our attention to the case $J_m = \mathbb{N}_m$ only (i.e., we will assume that $I_m = \varnothing$). In this case, we require that the following growth assumptions hold:

$$\begin{cases} |f_i(x, t, s_1, ..., s_m)| \leq C_f\left(\sum_{i\in\mathbb{N}_m} |s_i|^{\theta_i} + 1\right), \\ |g_i(x, t, s_1, ..., s_m)| \leq C_g\left(\sum_{i\in\mathbb{N}_m} |s_i|^{\beta_i} + 1\right), \end{cases} \tag{3.11}$$

for some $\theta_i, \beta_i > 0$ and some positive constants C_f, C_g.

Theorem 3.2. *Let (3.11) hold, and assume that $\exists \alpha_i, \sigma_i > 0$ such that*

$$\alpha_i |\overrightarrow{s}|^{p_i} \leq a_i(x, t, \overrightarrow{s}) \leq \sigma_i |\overrightarrow{s}|^{p_i}, \ i \in \mathbb{N}_m, \tag{3.12}$$

for any $\overrightarrow{s} = (s_1, ..., s_m) \in \mathbb{R}^m$. Let

$$\delta := \max_{i\in\mathbb{N}_m}\left\{2, \theta_i + 1, \frac{p_i}{2} + 1\right\}, \ \gamma := \max_{i\in\mathbb{N}_m}\left\{2, \beta_i + 1, \frac{p_i}{2} + 1\right\}. \tag{3.13}$$

Suppose that the system (1.1)-(1.4) satisfies the property $\boldsymbol{P}(\delta,\gamma)$-(i) (respectively, (ii) or (iii)), then it also satisfies $\boldsymbol{P}(\infty,\infty)$-(i) (respectively, (ii) or (iii)). In particular, for every $i\in\mathbb{N}_m$ and $T,\tau>0$ such that $T-2\tau>0$, the following estimate holds:

$$\begin{aligned}(3.14)\qquad &\sup_{(x,t)\in[T-\tau,T]\times\overline{\Omega}}\left|u_i(x,t)\right| \\ &\le Q\left(1+\|\overrightarrow{u}\|_{L^{\delta}([T-2\tau,T]\times\Omega)}+\|\overrightarrow{u}\|_{L^{\gamma}([T-2\tau,T]\times\Gamma)}\right),\end{aligned}$$

for some positive function Q which is independent of $\overrightarrow{u}$, time and the initial data. The function Q can be computed explicitly in terms of the physical paramaters of the problem.

We will now show how to deduce the property $\mathbf{P}(s_1,s_2)$, for some finite $s_1,s_2\ge1$, for the problem (1.1)-(1.2) subject to a dynamic boundary condition of the form (1.4). We shall first consider a special case. Let $\Gamma\subset\mathbb{R}^{N-1}$ consists of two disjoint open subsets Γ_1 and Γ_2, each $\overline{\Gamma}_i\backslash\Gamma_i$ is a S-null subset of Γ and $\Gamma=\overline{\Gamma}_1\cup\overline{\Gamma}_2$, such that $u_i,\ i\in\mathbb{N}_m$ satisfy (1.3) on $\Gamma_1\times(0,\infty)$, and

$$(3.15)\qquad u_i=0,\text{ on }\Gamma_2\times(0,\infty),\text{ for }i\in\mathbb{N}_m.$$

We assume that Γ_2 is a set of positive surface measure, and that the nonlinearities f_i,g_i satisfy the following special form of (3.6), that is,

$$(3.16)\qquad \begin{cases}\sum_{i\in\mathbb{N}_m}f_i(x,t,s_1,...,s_m)\,s_i\left|s_i\right|^{m_i-2}\ge-\sum_{i\in\mathbb{N}_m}C_{f_i}\left|s_i\right|^{m_i}-\widetilde{C}_f,\\ \sum_{i\in\mathbb{N}_m}g_i(x,t,s_1,...,s_m)\,s_i\left|s_i\right|^{m_i-2}\ge-\sum_{i\in\mathbb{N}_m}C_{g_i}\left|s_i\right|^{m_i}-\widetilde{C}_g,\end{cases}$$

for some $m_i\ge1$, and some positive constants C_{f_i},C_{g_i} and $\widetilde{C}_f,\widetilde{C}_g\ge0$. Note that (3.6) is equivalent to (3.16) for $m_i=2$.

The second main result gives a $\mathcal{X}^{\overrightarrow{r}}$-dissipative estimate for solutions of (1.1), (1.3), (1.4), (3.15).

Theorem 3.3. *Suppose Γ_2 is a set of positive surface measure. Let the assumptions (3.5), (3.16) be satisfied, and let $p_i>0$ for all $i\in\mathbb{N}_m$. Then, the system (1.1), (1.3), (1.4), (3.15) satisfies property $\boldsymbol{P}(\overrightarrow{r})$-(i) for $\overrightarrow{r}=(r_1,...,r_m)$ with $r_i=m_i$, $i\in\mathbb{N}_m$. Moreover, if $\overrightarrow{u}_0\in\mathcal{X}^{\overrightarrow{r}}$, then there also exists a positive function Q, independent of initial data and time, such that*

$$\sup_{t\ge0}\|\overrightarrow{u}(t)\|_{\mathcal{X}^{\overrightarrow{r}}}\le Q\left(\|\overrightarrow{u}_0\|_{\mathcal{X}^{\overrightarrow{r}}}\right).$$

Remark 3.2. Theorem 3.3 *only* holds if $\Gamma\neq\Gamma_1$, i.e., when the boundary Γ_2 has positive measure (cf. also Remark 4.2 below). We require different arguments for the case when $\Gamma_2\equiv\varnothing$.

We shall now derive another dissipative estimate for solutions of (1.1)-(1.4) in $\mathcal{X}^{\overrightarrow{r}}$-norm which also covers Dirichlet boundary conditions (for $i\in I_m$) and applies to the case when $\Gamma_2\equiv\varnothing$, without enforcing any further sign restrictions on all the p_i's (compare with the assumptions in Theorem 3.3). Analogous to (3.16), we shall assume that the functions f_i,g_i satisfy

$$(3.17)\qquad \begin{cases}\sum_{i\in\mathbb{N}_m}f_i(x,t,s_1,...,s_m,\zeta_i)\,s_i\left|s_i\right|^{m_i-2}\ge-\sum_{i\in\mathbb{N}_m}C_{f_i}\left|s_i\right|^{m_i+p_i}-\widetilde{C}_f,\\ \sum_{i\in\mathbb{N}_m}g_i(x,t,s_1,...,s_m)\,s_i\left|s_i\right|^{m_i-2}\ge-\sum_{i\in\mathbb{N}_m}C_{g_i}\left|s_i\right|^{m_i+p_i}-\widetilde{C}_g,\end{cases}$$

for some $m_i > 1$, and some *real* constants C_{f_i}, C_{g_i} and $\widetilde{C}_f, \widetilde{C}_g \geq 0$. Moreover, consider the (self-adjoint) eigenvalue problem for so-called Wentzell Laplacians $\Delta_{W,i}$ (see [22, Appendix]), as follows:

$$-a_i \Delta \varphi_i - C_{f_i} \varphi_i = \Lambda_i \varphi \text{ in } \Omega, \tag{3.18}$$

where

$$a_i := \alpha_i (m_i - 1) \left(\frac{2}{m_i + p_i} \right)^2,$$

with a boundary condition that depends on the eigenvalue Λ_i explicitly,

$$a_i \partial_{\mathbf{n}} \varphi_i - C_{g_i} \varphi_i = \Lambda_i \varphi_i \text{ on } \Gamma_1, \tag{3.19}$$

such that

$$\varphi_i = 0 \text{ on } \Gamma_2. \tag{3.20}$$

Here Γ_2 is assumed to be a set of nonnegative surface measure (the case $\Gamma_2 = \varnothing$ may be also allowed). Our second condition on the nonlinearities f_i, g_i is concerned with some sign assumptions on C_{f_i} and C_{g_i}. In particular, we assume that

$$\Lambda_1 := \inf_{i \in \mathbb{N}_m} \Lambda_{1,i} > 0. \tag{3.21}$$

Observe that, if $C_{f_i} < 0$, $C_{g_i} < 0$, for all $i \in \mathbb{N}_m$, then the first eigenvalue $\Lambda_{1,i}$ of (3.18)-(3.20) is always positive, for any $i \in \mathbb{N}_m$. Otherwise, we note that even when at least one of C_{f_i} or C_{g_i} is positive, it may still happen that (3.21) holds. In this sense, our system (1.1)-(1.4) becomes dissipative even when at least one of the terms f_i, g_i has the wrong sign at infinity, but the bad sign is compensated by the other term (see, also [31] for similar assumptions).

Theorem 3.4. *Suppose that Γ_2 is a set of nonnegative surface measure. Let (3.5), (3.17), (3.21) hold, and assume that $p_i > 0$ for all $i \in \mathbb{N}_m$. Then, the system (1.1), (1.4), (3.15) satisfies property $\boldsymbol{P}(\overrightarrow{r})$-(i), for $\overrightarrow{r} = (r_1, ..., r_m)$ with $r_i = m_i$, $i \in \mathbb{N}_m$. On the other hand, if $p_i = 0$ for some $i \in \mathbb{N}_m$, and if $\overrightarrow{u}_0 \in \mathcal{X}^{\overrightarrow{r}}$, then this system satisfies $\boldsymbol{P}(\overrightarrow{r})$-(ii) instead.*

Remark 3.3. All the above results can be extended to models which also incorporate advection effects in the domain Ω and on the boundary Γ (i.e., the reaction terms f_i, g_i and h_i may also depend on ∇u_i and $\nabla_\Gamma u_i$, respectively). Indeed, by making similar assumptions to (3.6)-(3.17), any integral over $|\nabla u_i|$ may be absorbed by diffusion in the bulk Ω with an appropiate application of suitable Young and Sobolev inequalities (cf. Section 4). We will return to these questions elsewhere.

4. Proof of main results

4.1. **Proof of Theorem 3.1.** In order to justify our computations, we need to construct an approximation scheme which relies on the existence of classical (smooth) solutions to the non-degenerate analogue of (1.1)-(1.4) (if at least one $p_i \neq 0$). One of the advantages of this construction is that now every (possibly, very weak) solution can be approximated by regular ones and the justification of our estimates for such solutions is immediate. To this end, for each $\epsilon > 0$, let us consider the following non-degenerate parabolic system:

$$\partial_t u_i - \operatorname{div}\left(a_i^\epsilon (u_i) \nabla u_i\right) + f_i \left(x, t, \overrightarrow{u}\right) = 0, \text{ in } \Omega \times (0, \infty), \tag{4.1}$$

for $i=1,...,m$, where $a_i^{\epsilon}(s_i):=a_i(s_i+\epsilon)\geq O(\epsilon)>0$, subject to the following set of boundary conditions

$$\partial_{\mathbf{n}}u_i+h_i(x,t,\overrightarrow{u})=0,\ \text{on }\Gamma\times(0,\infty),\ \ i\in I_m \tag{4.2}$$

and

$$\delta_i\partial_t u_i+a_{\epsilon}(u_i)\,\partial_{\mathbf{n}}(u_i)+g_i(x,t,\overrightarrow{u})=0,\ \text{on }\Gamma\times(0,\infty),\ \ i\in J_m. \tag{4.3}$$

We equip the system (1.1)-(1.3) with the initial conditions

$$u_{i|t=0}^{\epsilon}=u_{i0}^{\epsilon}\text{ in }\Omega,\ u_{i|t=0}^{\epsilon}=u_{i0}^{\epsilon}\text{ on }\Gamma, \tag{4.4}$$

for $u_i^{\epsilon}(0):=u_{i0}^{\epsilon}\in C^{\infty}\left(\overline{\Omega}\right),\ i\in\mathbb{N}_m$, such that

$$u_i^{\epsilon}(0)\to u_{i0}\text{ in }L^s(\Omega),\ u_i^{\epsilon}(0)_{|\Gamma}\to v_{i0}\text{ in }L^s(\Gamma),$$

for some given $s\geq 1$. Then, the approximate problem (4.1)-(4.4) admits a unique (smooth) classical solution with

$$\overrightarrow{u}^{\epsilon}=(u_1^{\epsilon},...,u_m^{\epsilon})\in C^1\left([0,t_*];\left(C^{\infty}\left(\overline{\Omega}\right)\right)^m\right),$$

for some $t_*>0$ and each $\epsilon>0$ (see [16, 17, 8, 31]). Being pedants, we cannot apply the main results of [16, 31] (cf. also [17]) directly to equations (4.1)-(4.4) since the functions a_i^{ϵ}, f_i, g_i and h_i are not smooth enough. Moreover, the solutions constructed this way may only exists locally in time for some interval $[0,t_*]$. However, by approximating the functions a_i^{ϵ}, f_i, g_i, h_i by smooth ones, say, in $C^{\infty}(\mathbb{R},\mathbb{R})$, we may apply Remark 4.1 below for the solutions of the approximate equations (4.1)-(4.4), and deduce the existence of a globally well-defined solution on $[0,T]$, for any $T>0$. Indeed, an apriori global bound in Hölder-norm for $\overrightarrow{u}^{\epsilon}$ guarantees the global existence of classical solutions (see, e.g., [31]). Nevertheless, even when these bounds are not available apriori, we may still choose to work with locally-defined in $[0,t_*]$, smooth solutions $\overrightarrow{u}^{\epsilon}$, that are globally defined on $\mathbb{R}_+$ in the lower-order L^p-norms; this turns out to be sufficient for our purposes. Indeed, if the solution $\overrightarrow{u}^{\epsilon}$ is globally-defined on $[0,T]$ in $\mathcal{X}^r$-norm, then it will also be global in $\mathcal{X}^{\infty}$-norm by (iii). As we shall see in this section, a global bound in $\mathcal{X}^r$-norm can be established for the same assumptions (3.5)-(3.17) on the nonlinearities.

We begin with the proof of Theorem 3.1, by following similar arguments to [11, 13] for the system (1.1) with static boundary conditions. From now on, c will denote a positive constant that is independent of t, ϵ, n, $\overrightarrow{u}$ and initial data, which only depends on the other structural parameters of the problem. Such a constant may vary even from line to line. Moreover, we shall denote by $Q_{\tau}(n)$ a monotone nondecreasing function in n of order τ, for some nonnegative constant τ, independent of n. More precisely, $Q_{\tau}(n)\sim cn^{\tau}$ as $n\to+\infty$. We begin by showing that the $\mathcal{X}^n$-norm of $\overrightarrow{u}=\overrightarrow{u}^{\epsilon}$ satisfies a local recursive relation which can be used to perform an iterative argument. We divide the proof of Theorem 3.1 into several steps.

Step 1 (The basic energy estimate in $\mathcal{X}^{n+1}$). We multiply (4.1) by $|u_i|^{n-1} u_i$, $n \geq 1$, and integrate over Ω, for each $i \in \mathbb{N}_m$. We obtain

$$\begin{aligned} &\frac{1}{(n+1)} \frac{d}{dt} \|u_i\|^{n+1}_{L^{n+1}(\Omega)} + \left\langle f_i\left(x,t,\overrightarrow{u}\right), |u_i|^{n-1} u_i \right\rangle_{L^2(\Omega)} \\ &+ n \int_\Omega a_i^\epsilon\left(u_i\right) |\nabla u_i|^2 |u_i|^{n-1} dx \\ &= \int_\Gamma a_i^\epsilon\left(u_i\right) \partial_{\mathbf{n}} u_i |u_i|^{n-1} u_i dS. \end{aligned} \tag{4.5}$$

Similarly, we multiply (4.2) and (4.3) by $|u_i|^{n-1} u_i$ and integrate each relation over Γ. We have

$$\begin{aligned} &\frac{\delta_i}{(n+1)} \frac{d}{dt} \|u_i\|^{n+1}_{L^{n+1}(\Gamma)} + \int_\Gamma a_i^\epsilon\left(u_i\right) \partial_{\mathbf{n}} u_i |u_i|^{n-1} u_i dS \\ &+ \left\langle g_i\left(x,t,\overrightarrow{u}\right), |u_i|^{n-1} u_i \right\rangle_{L^2(\Gamma)} \\ &= 0, \end{aligned} \tag{4.6}$$

for each $i \in J_m$, and

$$\int_\Gamma a_i^\epsilon\left(u_i\right) \partial_{\mathbf{n}} u_i |u_i|^{n-1} u_i dS + \left\langle a_i^\epsilon\left(u_i\right) h_i\left(x,t,\overrightarrow{u}\right), |u_i|^{n-1} u_i \right\rangle_{L^2(\Gamma)} = 0,\ i \in I_m. \tag{4.7}$$

In the case when (1.2) is replaced by a Dirichlet boundary condition for u_i, $i \in I_m$, equation (4.7) still holds since $h_i \equiv 0$ in that case.

Let us first observe that, in light of assumption (3.5), we have $a_i^\epsilon\left(s_i\right) \geq \alpha_i$, $\forall s_i \in \mathbb{R}$, $\epsilon > 0$ (recall that $p_i = 0$), which immediately implies

$$\int_\Omega a_i^\epsilon\left(u_i\right) |\nabla u_i|^2 |u_i|^{n-1} dx \geq \alpha_i \int_\Omega |\nabla u_i|^2 |u_i|^{n-1} dx,\ i \in \mathbb{N}_m. \tag{4.8}$$

Moreover, on account of of the assumptions (3.6) for f_i, g_i and a basic application of Hölder and Young inequalities, we deduce

$$\sum\nolimits_{i \in \mathbb{N}_m} \left\langle f_i\left(x,t,\overrightarrow{u}\right), |u_i|^{n-1} u_i \right\rangle_{L^2(\Omega)} \geq -c \sum\nolimits_{i \in \mathbb{N}_m} \|u_i\|^{n+1}_{L^{n+1}(\Omega)} - c, \tag{4.9}$$

and

$$\sum\nolimits_{i \in J_m} \left\langle g_i\left(x,t,\overrightarrow{u}\right), |u_i|^{n-1} u_i \right\rangle_{L^2(\Gamma)} \geq -c \sum\nolimits_{i \in J_m} \delta_i \|u_i\|^{n+1}_{L^{n+1}(\Gamma)} - c. \tag{4.10}$$

In order to estimate the source terms involving h_i on the boundary (4.7), we need the following lemma which allows us to control surface integrals in terms of volume integrals (see, e.g., [23]).

Lemma 4.1. *Let $n \geq 1$, $p \geq 0$, $s > -1$. Then for every $\varepsilon > 0$, there holds*

$$\int_\Gamma |u|^{s+n} dS \leq \varepsilon\left(s+n\right) \int_\Omega |\nabla u|^2 |u|^{p+n-1} dx + \frac{C}{\varepsilon}\left(s+n\right) \left(\|u\|^{s+n}_{L^{s+n}(\Omega)} + 1\right),$$

for some positive constant $C = C\left(p,s\right)$ independent of u, ε and n.

Applying Lemma 4.1 to $u = u_i$, with $s = 1$ and $p = 0$, $i \in I_m$, we have

$$\begin{aligned} \int_\Gamma |u_i|^{1+n} dS \leq{}& \varepsilon\left(n+1\right) \int_\Omega |\nabla u_i|^2 |u|^{n-1} dx \\ &+ \frac{C}{\varepsilon}\left(n+1\right) \left(\|u_i\|^{n+1}_{L^{n+1}(\Omega)} + 1\right). \end{aligned} \tag{4.11}$$

Thus, in light of the third assumption of (3.6), using (3.5), we find

$$\begin{aligned}(4.12)\qquad &\sum\nolimits_{i\in I_m}\left\langle a_i^{\epsilon}\left(u_i\right)h_i\left(x,t,\overrightarrow{u}\right),\left|u_i\right|^{n-1}u_i\right\rangle_{L^2(\Gamma)}\\ &\geq-\sum\nolimits_{i\in I_m}\left(\alpha_i C_{h_i}\left\|u_i\right\|_{L^{n+1}(\Gamma)}^{n+1}+\widetilde{C}_{h_i}\left\|u_i\right\|_{L^{n-1}(\Gamma)}^{n-1}\right)\\ &\geq-c\sum\nolimits_{i\in I_m}\alpha_i\left\|u_i\right\|_{L^{n+1}(\Gamma)}^{n+1}-c\end{aligned}$$

which can be bounded below by

$$\begin{aligned}(4.13)\qquad &-c\varepsilon\sum\nolimits_{i\in I_m}\alpha_i\left(n+1\right)\int_\Omega\left|\nabla u_i\right|^2\left|u_i\right|^{n-1}dx\\ &-\frac{c}{\varepsilon}\sum\nolimits_{i\in I_m}\left(n+1\right)\left(\left\|u_i\right\|_{L^{n+1}(\Omega)}^{n+1}+1\right),\end{aligned}$$

exploiting the estimate (4.11). Choose now $\varepsilon=\varepsilon_n>0$ in (4.13) such that

$$(4.14)\qquad \varepsilon_n=\max_{i\in\mathbb{N}_m}\frac{cn\alpha_i}{2\left(n+1\right)},\ \forall n\geq1,$$

and note that $\varepsilon_n\leq c$, uniformly as $n\to\infty$. Summing the equations (4.5) (respectively, (4.6) and (4.7)) over the sets $i\in\mathbb{N}_m$ (respectively, $i\in J_m$ and $i\in I_m$), then adding the relations that we obtain, on account of (4.9)-(4.10), (4.12)-(4.13) we deduce

$$\begin{aligned}(4.15)\qquad &\frac{d}{dt}\left\|\overrightarrow{u}\right\|_{\mathcal{X}^{n+1}}^{n+1}+cn\left(n+1\right)\sum\nolimits_{i\in\mathbb{N}_m}\int_\Omega\left|\nabla u_i\right|^2\left|u_i\right|^{n-1}dx\\ &\leq Q_2\left(n\right)\left(\left\|\overrightarrow{u}\right\|_{\mathcal{X}^{n+1}}^{n+1}+1\right),\end{aligned}$$

for any $n\geq1$. Here, the function $Q_2\left(n\right)\sim n^2$ as $n\to\infty$.

Step 2 (The local relation). Set $n_k=2^k-1$, $k\geq0$, and define

$$(4.16)\qquad \mathcal{Y}_k\left(t\right):=\left\|\overrightarrow{u}\left(t\right)\right\|_{\mathcal{X}^{n_k+1}}^{n_k+1},$$

for all $k\geq0$. Then, using the basic identity for $u=u_i$,

$$(4.17)\qquad \int_\Omega\left|\nabla u\right|^2\left|u\right|^{n-1}dx=\left(\frac{2}{n+1}\right)^2\int_\Omega\left|\nabla\left|u\right|^{\frac{n+1}{2}}\right|^2dx,$$

from (4.15) it holds

$$(4.18)\qquad \frac{d}{dt}\mathcal{Y}_k\left(t\right)+\sum\nolimits_{i\in\mathbb{N}_m}\gamma_{n_k}\int_\Omega\left|\nabla\left|u_i\right|^{\frac{n_k+1}{2}}\right|^2dx\leq Q_2\left(n_k\right)\left(\mathcal{Y}_k\left(t\right)+1\right),$$

for all $k\geq0$, where $0<\gamma_0\leq\gamma_{n_k}\sim c$, as $n_k\to\infty$. Let t,μ be two positive constants such that $t-\mu/n_k>0$. Their precise values will be chosen later. We claim that

$$(4.19)\qquad \mathcal{Y}_k\left(t\right)\leq M_k\left(t,\mu\right):=c_0\left(\mu\right)\left(n^k\right)^\sigma\left(\sup_{s\geq t-\mu/n_k}\mathcal{Y}_{k-1}\left(s\right)+1\right)^{\theta_k},\ \forall k\geq1,$$

where c_0, σ are positive constants independent of k, and $\theta_k\geq1$ is a bounded sequence for all k. The constant $c_0\left(\mu\right)$ is bounded if μ is bounded away from zero.

We will now prove (4.19) when $2<N$. The case $N\leq2$ requires only minor adjustments. We will follow an argument similar to the proof of [22, Theorem 2.3] (cf. also [23]). For each $k\geq0$, we define

$$r_{i,k}:=\frac{N\left(n_k+1\right)-\left(N-2\right)\left(1+n_k\right)}{N\left(n_k+1\right)-\left(N-2\right)\left(1+n_{k-1}\right)},\ s_{i,k}:=1-r_{i,k}.$$

We aim to estimate the term on the right-hand side of (4.15) in terms of the $\mathcal{X}^{1+n_{k-1}}$-norm of $\overrightarrow{u}$. First, Hölder and Sobolev inequalities (with the equivalent norm of Sobolev spaces in $W^{1,2}(\Omega)\subset L^{p_s}(\Omega)$, $p_s=2N/(N-2)$) yield

$$\begin{aligned}\int_\Omega |u_i|^{1+n_k}dx &\le \left(\int_\Omega |u_i|^{\frac{(n_k+1)N}{N-2}}dx\right)^{s_{i,k}}\left(\int_\Omega |u_i|^{1+n_{k-1}}dx\right)^{r_{i,k}} \\ &\le c\left(\int_\Omega \left|\nabla |u_i|^{\frac{(n_k+1)}{2}}\right|^2 dx+\int_\Omega |u_i|^{1+n_k}dx\right)^{\overline{s}_{i,k}} \\ &\quad\times\left(\int_\Omega |u_i|^{1+n_{k-1}}dx\right)^{r_{i,k}},\end{aligned} \tag{4.20}$$

with $\overline{s}_{i,k}:=s_{i,k}N/(N-2)\in(0,1)$. Applying Young's inequality on the right-hand side of (4.20), we get

$$\int_\Omega |u_i|^{1+n_k}dx \le \frac{\gamma_{n_k}}{4}\int_\Omega\left|\nabla |u_i|^{\frac{n_k+1}{2}}\right|^2dx+Q_{\tau_1}(n_k)\left(\int_\Omega |u_i|^{1+n_{k-1}}dx\right)^{z_{i,k}}, \tag{4.21}$$

for some positive constant τ_1 independent of n_k, and where

$$z_{i,k}:=r_{i,k}/(1-\overline{s}_{i,k})\ge 1$$

is bounded for all k. Note that we can choose τ_2 to be some fixed positive number since Q_{τ_2} also depends on $\gamma_{n_k}\sim c$.

To treat the boundary terms on the right-hand side of (4.15), we define for $k\ge 0$,

$$y_{i,k}:=\frac{(N-1)(n_k+1)-(N-2)(1+n_k)}{(N-1)(n_k+1)-(N-2)(1+n_{k-1})},\ x_{i,k}:=1-y_{i,k}.$$

On account of Hölder and Sobolev inequalities (e.g., $W^{1,2}(\Omega)\subset L^{q_s}(\Gamma)$, $q_s=2(N-1)/(N-2)$), we obtain

$$\begin{aligned}\int_\Gamma |u_i|^{1+n_k}dS &\le c\left(\int_\Gamma |u_i|^{\frac{(N-1)(n_k+1)}{N-2}}dS\right)^{x_{i,k}}\left(\int_\Gamma |u_i|^{1+n_{k-1}}dS\right)^{y_{i,k}} \\ &\le c\left(\int_\Omega\left|\nabla |u_i|^{\frac{(n_k+1)}{2}}\right|^2dx+\int_\Omega |u_i|^{1+n_k}dx\right)^{\overline{x}_{i,k}} \\ &\quad\times\left(\int_\Gamma |u_i|^{1+n_{k-1}}dS\right)^{y_{i,k}},\end{aligned} \tag{4.22}$$

with $\overline{x}_{i,k}:=x_{i,k}(N-1)/(N-2)$. Since $\overline{x}_{i,k}\in(0,1)$, we can apply Young's inequality on the right-hand side of (4.22), use the estimate for the $L^{1+n_k}(\Omega)$-norm of u_i from (4.20) in order to deduce the following estimate:

$$\int_\Gamma |u_i|^{1+n_k}dS\le\frac{\gamma_{n_k}}{4}\int_\Omega\left|\nabla |u_i|^{\frac{n_k+1}{2}}\right|^2dx+Q_{\tau_2}(n_k)\left(\int_\Omega |u_i|^{1+n_{k-1}}dx\right)^{l_{i,k}}, \tag{4.23}$$

for some positive constant τ_2 depending on τ_1, but which is independent of n_k, and where

$$l_{i,k}:=\frac{y_{i,k}}{(1-\overline{x}_{i,k})}\ge 1$$

is bounded for all $k\ge 0$. Inserting estimates (4.21)-(4.23) on the right-hand side of (4.18), we obtain the following inequality:

$$\partial_t\mathcal{Y}_k(t)+\sum\nolimits_{i\in\mathbb{N}_m}\gamma_{n_k}\int_\Omega\left|\nabla |u_i|^{\frac{n_k+1}{2}}\right|^2dx\le c(n_k)^{\sigma_1}(\mathcal{Y}_{k-1}+1)^{\theta_k}, \tag{4.24}$$

where c, σ_1 are positive constants independent of k, and

$$\theta_k := \max(\max_i \{z_{i,k}\}, \max_i \{l_{i,k}\}) \geq 1$$

is a bounded sequence for all k.

We are now ready to prove (4.19) using (4.24). To this end, let $\zeta(s)$ be a positive function $\zeta : \mathbb{R}_+ \to [0,1]$ such that $\zeta(s) = 0$ for $s \in [0, t-\mu/n_k]$, $\zeta(s) = 1$ if $s \in [t, +\infty)$ and $|d\zeta/ds| \leq n_k/\mu$, if $s \in (t-\mu/n_k, t)$. We define $Z_k(s) = \zeta(s)\mathcal{Y}_k(s)$ and notice that

$$\frac{d}{ds} Z_k(s) \leq \frac{n_k}{\mu}\mathcal{Y}_k(s) + \zeta(s)\frac{d}{ds}\mathcal{Y}_k(s).$$

Combining this estimate with (4.18), (4.21), (4.23) and noticing that $Z_k \leq \mathcal{Y}_k$, we deduce the following estimate for Z_k:

$$\frac{d}{ds} Z_k(s) + C(\mu) n_k Z_k(s) \leq M_k(t,\mu), \text{ for all } s \in [t-\mu/n_k, +\infty), \tag{4.25}$$

for some positive constant C independent of k. Integrating (4.25) with respect to s from $t-\mu/n_k$ to t and taking into account the fact that $Z_k(t-\mu/n_k) = 0$, we obtain that

$$\mathcal{Y}_k(t) = Z_k(t) \leq M_k(t,\mu)\left(1-e^{-C\mu}\right),$$

which proves the claim (4.19).

Step 3 (The iterative argument). Let now $\tau^{'} > \tau > 0$ be given; define $\mu = 2(\tau^{'} - \tau)$, $t_0 = \tau^{'}$ and $t_k = t_{k-1} - \mu/n_k$, $k \geq 1$. Using (4.19), we have

$$\sup_{t \geq t_{k-1}} \mathcal{Y}_k(t) \leq c_0 (n_k)^{\sigma} (\sup_{s \geq t_k} \mathcal{Y}_{k-1}(s) + 1)^{\theta_k}, \ k \geq 1. \tag{4.26}$$

Here $c_0 = c_0(\mu)$ depends only on μ. Now let us define

$$\overline{C} := \sup_{s \geq t_1 = \tau} (\mathcal{Y}_0(s) + 1) = \sup_{s \geq t_1 = \tau} (\|\overrightarrow{u}(s)\|_{\mathcal{X}^1} + 1). \tag{4.27}$$

Thus, we can iterate in (4.26) with respect to $k \geq 1$ and obtain that

(4.28)

$$\begin{aligned}\sup_{t \geq t_{k-1}} \mathcal{Y}_k(t) &\leq (c_0 n_k^{\sigma})\left(c_0 n_{k-1}^{\sigma}\right)^{\theta_k}\left(c_0 n_{k-2}^{\sigma}\right)^{\theta_k\theta_{k-1}} \cdot \ldots \cdot (c_0 n_0^{\sigma})^{\theta_k\theta_{k-1}\ldots\theta_0} (\overline{C})^{\xi_k} \\ &\leq c_0^{A_k} 2^{\sigma B_k} \left(\overline{C}\right)^{\xi_k},\end{aligned}$$

where $\xi_k := \theta_k\theta_{k-1}\ldots\theta_0$, and

$$A_k := 1 + \theta_k + \theta_k\theta_{k-1} + \ldots + \theta_k\theta_{k-1}\ldots\theta_0, \tag{4.29}$$

$$B_k := k + \theta_k(k-1) + \theta_k\theta_{k-1}(k-2) + \ldots + \theta_k\theta_{k-1}\ldots\theta_0. \tag{4.30}$$

We can easily show that

$$A_k \leq (c_1 + n_k)\sum_{j=1}^{\infty}\frac{1}{c_1 + n_j} \text{ and } B_k \leq (c_2 + n_k)\sum_{j=1}^{\infty}\frac{j}{c_2 + n_j}, \tag{4.31}$$

for some positive constants c_1, c_2 independent of k, μ. Therefore, since

$$\sup_{t \geq t_0} \mathcal{Y}_k(t) \leq \sup_{t \geq t_{k-1}} \mathcal{Y}_k(t) \leq c_0^{A_k} 2^{\sigma B_k} \left(\overline{C}\right)^{\xi_k} \tag{4.32}$$

and the series in (4.31) are convergent, we can take the $1+n_k$-root on both sides of (4.32) and let $k \to +\infty$. We deduce

$$\sup_{t \geq t_0 = \tau'} \|\overrightarrow{u}(t)\|_{\mathcal{X}^\infty} \leq \lim_{k \to +\infty} \sup_{t \geq t_0} (\mathcal{Y}_k(t))^{1/(1+n_k)},$$

which, on account of (4.32), yields

$$\sup_{t \geq t_0 = \tau'} \|\overrightarrow{u}(t)\|_{\mathcal{X}^\infty} \leq C(\mu) \left(\overline{C}\right)^{1/c_3}, \tag{4.33}$$

for some positive constant c_3 independent of t, k, $\overrightarrow{u}$, ϵ, μ, and initial data. Note that $\overline{C}$ depends on τ (see (4.27)).

Step 4 (The final argument). Let us first assume that **Property P**(1)-(i) holds. Then we already know that the $\mathcal{X}^1$-norm of $\overrightarrow{u}(t)$ is bounded independently of the initial data, for each $t \geq \tau$. Therefore, from (4.33) we also obtain the claim for the $\mathcal{X}^\infty$-norm of $\overrightarrow{u}(t)$, i.e., **Property P**(∞)-(i) holds as well. If, on the other hand, **Property P**(1)-(ii) holds, we can choose $\tau' = \tau + 2\mu$ with $\mu = 1$ so that $\overline{C}$ and $C(\mu)$ are bounded uniformly with respect to initial data as $\tau \to \infty$. Hence, **Property P**(∞)-(ii) is also satisfied by letting $\tau \to \infty$ in (4.33). In order to show the final property (iii), taking advantage of the fact that the initial data $\overrightarrow{u}_0 \in \mathcal{X}^\infty$, it suffices to note that in place of the inequality (4.19), we may use instead the inequality

$$\mathcal{Y}_k(t) \leq Q(\|\overrightarrow{u}_0\|_{\mathcal{X}^\infty}, \sup_{t>0} M_k(t,\mu)),$$

which is an immediate consequence of (4.24). Arguing analogously as in [31, Lemma 5.5.3], we obtain the claim. The proof of Theorem 4.33 is now complete.

Remark 4.1. It was proven in [31, Section 5] that maximal L^p-regularity for (4.1)-(4.4) can be used to reduce the question of global existence of the solutions $\overrightarrow{u}^\epsilon$ in a space of maximal regularity, to the boundedness of $\overrightarrow{u}^\epsilon$ in a Hölder norm $C^{0,\beta}\left(\overline{\Omega}\right)$, $\beta > 0$. It should be possible to prove, under the natural assumptions of Theorem 3.1, that every classical solution of problem (4.1)-(4.4) is globally Hölder continuous on $\overline{\Omega}$. Establishing global Hölder continuity for solutions to systems with dynamic boundary conditions requires a more detailed analysis, involving careful local estimates of the solution near the boundary. Of course, as in the case of Dirichlet/Robin boundary conditions for (4.1) (see, e.g., [12]), these Hölder bounds should apriori depend on the L^∞-norm of the solution. Thus, our analysis constitutes only the first step in proving boundedness in Hölder norm $C^{0,\beta}\left(\overline{\Omega}\right)$. This question remains open for now.

4.2. **Proof of Theorem 3.2.** We shall divide the proof into several steps. As in Section 4.1, we can justify our computations by exploiting the approximation scheme (4.1)-(4.4). As before, c will denote a positive constant that is independent of t, ϵ, n, $\overrightarrow{u}$ and initial data, which only depends on the other structural parameters of the problem. Such a constant may vary even from line to line. Without loss of generality, we may assume that $\delta_i = 1$, for all $i \in J_m = \mathbb{N}_m$.

Let T, τ and L be positive numbers such that $T - 2\tau > 0$ and $L \geq 1$. We set $t_0 = T - 2\tau$ and define the sequences

$$t_n = t_{n-1} + \frac{\tau}{2^n},\ k_n = L\left(2 - \frac{1}{2^n}\right),\ \text{for all } n \geq 1.$$

Consider the (smooth) cut-off functions $\eta_n \in C^1(\mathbb{R}, [0,1])$ with the property that

$$\eta_n(t) = \begin{cases} 1, & t \geq t_n, \\ 0, & t < t_{n-1}. \end{cases}$$

Next, denote $Q_n := I_n \times \overline{\Omega}$, where $I_n := [t_{n-1}, T]$, and the sets

$$A_{i,n}^{\Omega} := \{(x,t) \in I_n \times \Omega : u_i(x,t) > k_n\}, \ A_{i,n}^{\Gamma} := \{(x,t) \in I_n \times \Gamma : u_i(x,t) > k_n\}.$$

Let $\overline{A}_{i,n} = A_{i,n}^{\Omega} \cup A_{i,n}^{\Gamma}$, and note that

$$\overline{A}_{i,n} = \{(x,t) \in Q_n : u_i(x,t) > k_n\}.$$

Finally, we denote by $\left|A_{i,n}^{\Omega}\right|$ the ($N+1$-dimensional) Lebesgue measure of the set $A_{i,n}^{\Omega}$, and by $\left|A_{i,n}^{\Gamma}\right|$, the ($N$-dimensional) Lebesgue measure of $A_{i,n}^{\Gamma}$, respectively. We note that, according to (3.2)-(3.3), we have

$$|Q_n| = |I_n| \, \mu\left(\overline{\Omega}\right) = |I_n| \left(|\Omega| + |\Gamma|\right),$$

and we can do so similarly for the set $\overline{A}_{i,n}$.

Step 1. (The energy inequality). We define the truncated functions

$$u_{i,n}(x,t) := \max\{u_i(x,t) - k_n, 0\} = (u_i - k_n)_+ .$$

We begin by multiplying equation (4.1) by $u_{i,n}\eta_n^2(t)$ and integrating the resulting identity over $I_n \times \Omega$. Then, we multiply (4.3) by $u_{i,n}\eta_n^2(t)$ and integrate over $I_n \times \Gamma$. Adding as usual (cf., e.g., Section 4.1), then exploiting the growth assumptions (3.11) on f_i and g_i, the fact that $\left|\eta_n^{'}(t)\right| \leq 2^n/\tau$, we obtain after standard transformations

(4.34)

$$\begin{aligned}
&\max_{t \in I_n}\left(\int_{\Omega} u_{i,n}^2(x,t)\,dx + \int_{\Gamma} u_{i,n}^2(x,t)\,dS\right) + \iint_{I_n \times \Omega} a_i(u_i)\left|\nabla\left(u_{i,n}\eta_n\right)(x,t)\right|^2 dxdt \\
&\leq c\iint_{A_{i,n}^{\Omega}} \left(\frac{2^n}{\tau}u_{i,n}^2(x,t)\,\eta_n + \sum\nolimits_{j\in\mathbb{N}_m}\left|u_j(x,t)\right|^{\theta} u_{i,n}(x,t)\,\eta_n^2(t) + u_{i,n}(x,t)\,\eta_n^2\right)dxdt \\
&+ c\iint_{A_{i,n}^{\Gamma}} \left(\frac{2^n}{\tau}u_{i,n}^2(x,t)\,\eta_n + \sum\nolimits_{j\in\mathbb{N}_m}\left|u_j(x,t)\right|^{\beta} u_{i,n}(x,t)\,\eta_n^2 + u_{i,n}(x,t)\,\eta_n^2\right)dSdt,
\end{aligned}$$

where we have set $\theta := \max_{i \in \mathbb{N}_m}\theta_i$ and $\beta := \max_{i\in\mathbb{N}_m}\beta_i$. Now, we wish to estimate the terms on the right-hand side of (4.34). To this end, set

$$\mathcal{A}_n^{\Omega} := \cup_{k\geq 1}A_{k,n}^{\Omega}, \ \mathcal{A}_n^{\Gamma} := \cup_{k\geq 1}A_{k,n}^{\Gamma}$$

and note that on $\mathcal{A}_n^{\Omega}\backslash A_{i,n}^{\Omega}$ and $\mathcal{A}_n^{\Gamma}\backslash A_{i,n}^{\Gamma}$, respectively, we have $u_i(x,t) \leq k_n \leq 2L$ and $u_i(x,t)_{|\Gamma} \leq k_n \leq 2L$, respectively. Therefore,

$$\iint_{\mathcal{A}_n^{\Omega}\backslash A_{i,n}^{\Omega}} |u_i|^{\theta+1}\,dxdt \leq cL^{\theta+1}\iint_{\mathcal{A}_n^{\Omega}\backslash A_{i,n}^{\Omega}} (1)\,dxdt \leq cL^{\theta+1}\sum\nolimits_{j\in\mathbb{N}_m}\left|A_{j,n}^{\Omega}\right|, \tag{4.35}$$

and, analogously, for the trace of u_i we have

(4.36)

$$\iint_{\mathcal{A}_n^{\Gamma}\backslash A_{i,n}^{\Gamma}} |u_i|^{\beta+1}\,dSdt \leq cL^{\beta+1}\iint_{\mathcal{A}_n^{\Gamma}\backslash A_{i,n}^{\Gamma}} (1)\,dSdt \leq cL^{\beta+1}\sum\nolimits_{j\in\mathbb{N}_m}\left|A_{j,n}^{\Gamma}\right|,$$

Since $k_n \sim L$, as $n \to \infty$, it is easy to see that the following inequalities hold:

$$\begin{cases} L^{\alpha+1} \left| A_{j,n}^{\Omega} \right| \leq c k_n^{\alpha+1} \left| A_{j,n}^{\Omega} \right| \leq c \iint_{A_{j,n}^{\Omega}} \left| u_j \right|^{\alpha+1} dxdt, \\ L^{\beta+1} \left| A_{j,n}^{\Gamma} \right| \leq c k_n^{\beta+1} \left| A_{j,n}^{\Gamma} \right| \leq c \iint_{A_{j,n}^{\Gamma}} \left| u_j \right|^{\beta+1} dSdt. \end{cases}$$

From these estimates, we thus find that

$$\begin{aligned} \iint_{I_n \times \Omega} \left| u_j \right|^{\theta} u_{i,n} dxdt &\leq \iint_{\mathcal{A}_n^{\Omega}} \left(\left| u_j \right|^{\theta+1} + \left| u_i \right|^{\theta+1} \right) dxdt \\ &\leq c \sum\nolimits_{j \in \mathbb{N}_m} \iint_{A_{j,n}^{\Omega}} \left| u_j \right|^{\theta+1} dxdt \end{aligned} \tag{4.37}$$

and, similarly,

$$\begin{aligned} \iint_{I_n \times \Gamma} \left| u_j \right|^{\beta} u_{i,n} dSdt &\leq \iint_{\mathcal{A}_n^{\Gamma}} \left(\left| u_j \right|^{\beta+1} + \left| u_i \right|^{\beta+1} \right) dxdt \\ &\leq c \sum\nolimits_{j \in \mathbb{N}_m} \iint_{A_{j,n}^{\Gamma}} \left| u_j \right|^{\beta+1} dSdt. \end{aligned} \tag{4.38}$$

Hence, using the above inequalities (4.37)-(4.38) on the right-hand side of (4.34), and summing the resulting relation over $i \in \mathbb{N}_m$, we deduce

(4.39)

$$\begin{aligned} &\max_{t \in I_n} \left(\sum\nolimits_{i \in \mathbb{N}_m} \int_{\Omega} u_{i,n}^2 (x,t)\, dx + \sum\nolimits_{i \in \mathbb{N}_m} \int_{\Gamma} u_{i,n}^2 (x,t)\, dS \right) \\ &+ \sum\nolimits_{i \in \mathbb{N}_m} \iint_{I_n \times \Omega} \alpha_i \left| u_i \right|^{p_i} \left| \nabla (u_{i,n} \eta_n) \right|^2 dxdt \\ &\leq \frac{2^n c}{\tau} \sum\nolimits_{i \in \mathbb{N}_m} \iint_{A_{i,n}^{\Omega}} \left| u_i (x,t) \right|^{\delta} dxdt + \frac{2^n c}{\tau} \sum\nolimits_{i \in \mathbb{N}_m} \iint_{A_{i,n}^{\Gamma}} \left| u_i (x,t) \right|^{\gamma} dSdt, \end{aligned}$$

where δ and γ are defined as in (3.13). Here we have also used assumption (3.5).
Step 2. (Additional estimates). From the definition of k_n, we see that $1 - k_n/k_{n+1} \geq 2^{-(n+2)}$, which yields

$$\begin{aligned} \iint_{A_{i,n+1}^{\Omega}} \left| u_i \right|^{\delta} dxdt &\leq 2^{(n+2)\delta} \iint_{A_{i,n+1}^{\Omega}} \left| u_i \right|^{\delta} \left(1 - \frac{k_n}{k_{n+1}} \right) dxdt \\ &\leq 2^{n\delta} c \iint_{A_{i,n+1}^{\Omega}} (u_i - k_n)_+^{\delta} dxdt. \end{aligned} \tag{4.40}$$

Moreover, the same argument gives

$$\iint_{A_{i,n+1}^{\Gamma}} \left| u_i \right|^{\gamma} dSdt \leq 2^{n\gamma} c \iint_{A_{i,n+1}^{\Gamma}} (u_i - k_n)_+^{\gamma} dSdt. \tag{4.41}$$

On the other hand, since on $A_{i,n+1}^{\Omega} \cup A_{i,n+1}^{\Gamma}$, we have $(u_i - k_n)_+ \geq k_{n+1} - k_n$, there holds

$$\begin{aligned} &\iint_{A_{i,n+1}^{\Omega}} (u_i - k_n)_+^{\delta} dxdt \geq (k_{n+1} - k_n)^{\delta} \left| A_{i,n+1}^{\Omega} \right| \geq c \frac{L^{\delta}}{2^{n\delta}} \left| A_{i,n+1}^{\Omega} \right|, \\ &\iint_{A_{i,n+1}^{\Gamma}} (u_i - k_n)_+^{\gamma} dSdt \geq (k_{n+1} - k_n)^{\gamma} \left| A_{i,n+1}^{\Gamma} \right| \geq c \frac{L^{\gamma}}{2^{n\gamma}} \left| A_{i,n+1}^{\Gamma} \right|. \end{aligned} \tag{4.42}$$

Because of these two inequalities (4.42), for any positive number λ such that, if $\lambda < \delta$ and $\lambda < \gamma$, on account of Holder's inequality, it also holds

$$\begin{aligned}\iint_{A^{\Omega}_{i,n+1}} (u_i - k_{n+1})_+^{\lambda}\, dxdt &\leq \left(\iint_{A^{\Omega}_{i,n+1}} (u_i - k_{n+1})_+^{\delta}\, dxdt\right)^{\lambda/\delta} \left|A^{\Omega}_{i,n+1}\right|^{1-\lambda/\delta} \\ &\leq \frac{c2^{n(\delta-\lambda)}}{L^{\delta-\lambda}} \iint_{A^{\Omega}_{i,n+1}} (u_i - k_n)_+^{\delta}\, dxdt,\end{aligned} \tag{4.43}$$

and

$$\begin{aligned}\iint_{A^{\Gamma}_{i,n+1}} (u_i - k_{n+1})_+^{\lambda}\, dSdt &\leq \left(\iint_{A^{\Gamma}_{i,n+1}} (u_i - k_{n+1})_+^{\gamma}\, dxdt\right)^{\lambda/\gamma} \left|A^{\Gamma}_{i,n+1}\right|^{1-\lambda/\gamma} \\ &\leq \frac{c2^{n(\gamma-\lambda)}}{L^{\gamma-\lambda}} \iint_{A^{\Gamma}_{i,n+1}} (u_i - k_n)_+^{\gamma}\, dSdt.\end{aligned} \tag{4.44}$$

Next, we will collect some useful inequalities which follow from the following well-known embeddings: $H^1(\Omega) \subset L^{p_s}$, $p_s := 2N/(N-2)$ and $H^1(\Omega) \subset L^{q_s}(\Gamma)$, $q_s := 2(N-1)/(N-2)$. We give the argument for $N > 2$, the case $N \leq 2$ can be treated analogously. Suppressing the dependance on the subscript n for the moment, from Hölder's inequality and these Sobolev embeddings, for every $v^{M_i} \in W^{1,2}(I \times \Omega)$ we have

$$\begin{aligned}\iint_{I\times\Omega} v^s dxdt &\leq \int_I \left(\int_\Omega v^{M_i p_s} dx\right)^{\frac{N-2}{N}} \left(\int_\Omega v^2 dx\right)^{\frac{2}{N}} dt \\ &\leq \left(\iint_{I\times\Omega} \left(\left|\nabla v^{M_i}\right|^2 + |v|^{M_i}\right) dxdt\right) \times \left(\max_{t\in I} \int_\Omega v^2 dx\right)^{\frac{2}{N}},\end{aligned} \tag{4.45}$$

where, for each $M_i > 0$, we have set

$$s = 2M_i + \frac{4}{N}.$$

Similarly, for each $M_i > 0$ and $l = 2M_i + 2/(N-1)$, we have

$$\begin{aligned}\iint_{I\times\Gamma} v^s dS &\leq \int_I \left(\int_\Gamma v^{M_i q_s} dS\right)^{\frac{N-2}{N-1}} \left(\int_\Gamma v^2 dS\right)^{\frac{1}{N-1}} dt \\ &\leq \left(\iint_{I\times\Omega} \left(\left|\nabla v^{M_i}\right|^2 + |v|^{M_i}\right) dxdt\right) \times \left(\max_{t\in I} \int_\Gamma v^2 dS\right)^{\frac{1}{N-1}}.\end{aligned} \tag{4.46}$$

Exploiting (4.45)-(4.46) with $M_i = p_i/2 + 1$, $I = I_{n+1}$, $v = (u_i - k_{n+1})_+$, we get

$$\begin{aligned}&\iint_{I_{n+1}\times\Omega} (u_i - k_{n+1})_+^s\, dxdt \\ &\leq \left(\iint_{I_{n+1}\times\Omega} \left(|u_i|^{p_i} \left|\nabla u_{i,n+1}\right|^2 + u_{i,n+1}^{M_i}\right) dxdt\right) \times \left(\max_{t\in I_{n+1}} \int_\Omega u_{i,n+1}^2 dx\right)^{\frac{2}{N}}\end{aligned} \tag{4.47}$$

and
(4.48)

$$\iint_{I_{n+1}\times\Gamma}(u_i-k_{n+1})_+^l\,dSdt$$
$$\leq\left(\iint_{I_{n+1}\times\Omega}\left(|u_i|^{p_i}\left|\nabla u_{i,n+1}\right|^2+u_{i,n+1}^{M_i}\right)dxdt\right)\times\left(\max_{t\in I_{n+1}}\int_\Gamma u_{i,n+1}^2dx\right)^{\frac{1}{N-1}}.$$

Finally, from (4.39) we see that estimates (4.40)-(4.41) yield the following inequality

$$\begin{aligned}(4.49)\qquad&\max_{t\in I_n}\left(\sum\nolimits_{i\in\mathbb{N}_m}\int_\Omega u_{i,n}^2\,(x,t)\,dx+\sum\nolimits_{i\in\mathbb{N}_m}\int_\Gamma u_{i,n}^2\,(x,t)\,dS\right)\\&+\sum\nolimits_{i\in\mathbb{N}_m}\iint_{I_n\times\Omega}\alpha_i\,|u_i|^{p_i}\left|\nabla\left(u_{i,n}\eta_n\right)\right|^2dxdt\\&\leq\frac{2^{n(\delta+1)}c}{\tau}\sum\nolimits_{i\in\mathbb{N}_m}\iint_{A_{i,n}^\Omega}(u_i-k_{n-1})_+^\delta\,dxdt\\&+\frac{2^{n(\gamma+1)}c}{\tau}\sum\nolimits_{i\in\mathbb{N}_m}\iint_{A_{i,n}^\Gamma}(u_i-k_{n-1})_+^\gamma\,dSdt.\end{aligned}$$

Step 3. (The iterative argument). We continue our main argument by first recalling the following result (see, e.g., [10, Lemma 4.1]).

Lemma 4.2. *Let $\{\mathcal{Y}_n\}$ be a sequence of positive numbers such that it satisfies*

$$\mathcal{Y}_{n+1}\leq Cb^n\mathcal{Y}_n^{1+\kappa},\tag{4.50}$$

for some constants $C,b,\kappa>0$. If $\mathcal{Y}_0\leq C^{-1/\kappa}b^{-1/\kappa^2}$, then $\mathcal{Y}_n\to0$ as $n\to\infty$.

Define

$$\mathcal{Y}_{i,n}:=\frac{1}{|Q_n|}\left(\iint_{I_n\times\Omega}(u_i-k_n)_+^\delta\,dxdt+\iint_{I_n\times\Gamma}(u_i-k_n)_+^\gamma\,dSdt\right),$$

where we recall that $Q_n=I_n\times\overline{\Omega}$ and

$$|Q_n|=|I_n\times\Omega|+|I_n\times\Gamma|\,.$$

Set $\mathcal{Y}_n=\sum_{i=1}^m\mathcal{Y}_{i,n}$. The goal now is to show that the sequence $\{\mathcal{Y}_n\}$ satisfies a recursive relation of the form (4.50). First, using the definition of $\mathcal{Y}_n$, we can rewrite (4.49) as the following inequality:

$$\begin{aligned}(4.51)\qquad&\max_{t\in I_{n+1}}\left(\sum\nolimits_{i\in\mathbb{N}_m}\int_\Omega u_{i,n+1}^2\,(x,t)\,dx+\sum\nolimits_{i\in\mathbb{N}_m}\int_\Gamma u_{i,n+1}^2\,(x,t)\,dS\right)\\&+\sum\nolimits_{i\in\mathbb{N}_m}\iint_{I_{n+1}\times\Omega}\alpha_i\,|u_i|^{p_i}\left|\nabla\left(u_{i,n+1}\eta_{n+1}\right)\right|^2dxdt\\&\leq c2^{(n+1)(\rho+1)}\tau^{-1}\,|Q_{n+1}|\,\mathcal{Y}_n,\end{aligned}$$

for $\rho:=\max(\gamma,\delta)>1$. Secondly, applying (4.43) to the bulk integral over $u_{i,n+1}^{M_i}$ (where $M_i:=p_i/2+1$), which occurs in the integrals in (4.47)-(4.48), and then using (4.51) to estimate the second terms in those products, we obtain

$$\begin{aligned}(4.52)\qquad&\iint_{I_{n+1}\times\Omega}(u_i-k_{n+1})_+^s\,dxdt\\&\qquad\leq c\left(\frac{2^{n\rho}}{\tau}+\frac{2^{n(\delta-M_i)}}{L^{\delta-M_i}}\right)|Q_{n+1}|\,\mathcal{Y}_n\left(\frac{c2^{n\rho}}{\tau}\,|Q_{n+1}|\,\mathcal{Y}_n\right)^{\frac{2}{N}},\end{aligned}$$

and

$$\iint_{I_{n+1}\times\Gamma}(u_i-k_{n+1})_+^l\,dSdt \le c\left(\frac{2^{n\rho}}{\tau}+\frac{2^{n(\gamma-M_i)}}{L^{\gamma-M_i}}\right)|Q_{n+1}|\,\mathcal{Y}_n\left(\frac{c2^{n\rho}}{\tau}|Q_{n+1}|\,\mathcal{Y}_n\right)^{\frac{1}{N-1}}. \tag{4.53}$$

Hölder's inequality applied to $\mathcal{Y}_{i,n+1}$ yields

$$\begin{aligned}\mathcal{Y}_{i,n+1} \le{}& \frac{1}{|Q_{n+1}|}\left(\iint_{I_{n+1}\times\Omega}(u_i-k_{n+1})_+^s\,dxdt\right)^{\delta/s}\left|A^{\Omega}_{i,n+1}\right|^{1-\delta/s}\\ &+\frac{1}{|Q_{n+1}|}\left(\iint_{I_{n+1}\times\Gamma}(u_i-k_{n+1})_+^l\,dSdt\right)^{\gamma/l}\left|A^{\Gamma}_{i,n+1}\right|^{1-\gamma/l}.\end{aligned} \tag{4.54}$$

Inserting the estimates for $\left|A^{\Omega}_{i,n+1}\right|$ and $\left|A^{\Gamma}_{i,n+1}\right|$, respectively, from (4.42), on the right-hand side of (4.54), we deduce

$$\begin{aligned}\mathcal{Y}_{i,n+1} \le{}& c\,|Q_{n+1}|^{\frac{2\delta}{Ns}}\,\mathcal{Y}_n^{1+\frac{2\delta}{Ns}}\left(\frac{2^{n\rho}}{\tau}+\frac{2^{n(\delta-M_i)}}{L^{\delta-M_i}}\right)^{\delta/s}\\ &\times\left(\frac{c2^{n\rho}}{\tau}\right)^{\frac{2\delta}{Ns}}\left(\frac{2^{n\delta}}{L^{\delta}}\right)^{1-\delta/s}\\ &+c\,|Q_{n+1}|^{\frac{\gamma}{(N-1)l}}\,\mathcal{Y}_n^{1+\frac{\gamma}{(N-1)l}}\left(\frac{2^{n\rho}}{\tau}+\frac{2^{n(\gamma-M_i)}}{L^{\gamma-M_i}}\right)^{\gamma/l}\\ &\times\left(\frac{c2^{n\rho}}{\tau}\right)^{\frac{\gamma}{(N-1)l}}\left(\frac{2^{n\gamma}}{L^{\gamma}}\right)^{1-\gamma/l}.\end{aligned}$$

Henceforth, by setting

$$\kappa:=\kappa(\delta,\gamma)=\begin{cases}\max\left(\frac{2\delta}{Ns},\frac{\gamma}{(N-1)l}\right), & \text{if } \mathcal{Y}_n\ge 1,\\ \min\left(\frac{2\delta}{Ns},\frac{\gamma}{(N-1)l}\right), & \text{if } \mathcal{Y}_n< 1,\end{cases}$$

the above inequality yields the recursive relation

$$\mathcal{Y}_{n+1}\le Cb^n\mathcal{Y}_n^{1+\kappa},\ \text{with } \kappa>0,$$

where $C\sim L^{-\sigma}$ depends on τ^{-1} and $b\sim 2^{\zeta}$, for some positive constants σ,ζ depending on $\delta,\gamma,s,\ l$. Therefore, if we choose $L\ge 1$ sufficiently large so there holds

$$\mathcal{Y}_0\le C^{-1/\kappa}b^{-1/\kappa^2}\lessapprox L^{\sigma/\kappa}b^{-1/\kappa^2},$$

then by Lemma 4.2, it follows that $\mathcal{Y}_n\to 0$ as $n\to\infty$. This implies that

$$\sup_{(x,t)\in[T-\tau,T]\times\overline{\Omega}}u_i(x,t)\le\lim_{n\to\infty}k_n\le 2L.$$

In order to estimate $u_i(x,t)$ from below it suffices to apply the result just obtained to the functions $\widetilde{u}_i(x,t)=-u_i(x,t)$, which satisfies a system of the same type as for $u_i(x,t)$, with nonlinearities $\widetilde{a}_i(x,t,\widetilde{u}_i)=-a_i(x,t,-u_i)$, $\widetilde{f}_i(x,t,\widetilde{u}_i)=-f_i(x,t,-u_i)$ and $\widetilde{g}_i(x,t,\widetilde{u}_i)=-g_i(x,t,-u_i)$, respectively. These functions are subject to the same conditions (3.12), (3.11). This yields the desired estimate (3.14). Finally, we may conclude that if T is sufficiently large, we can take $\tau=1$ in (3.14), which also immediately gives the first conclusion in the theorem. The proof is finished.

4.3. **Proof of Theorems 3.3, 3.4.** In order to justify our computations for problem (1.1), (1.3), (1.4), (3.15), it is not clear how to use the scheme introduced at the beginning of the section due to the nature of the boundary domain (indeed, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 \neq \varnothing$ in that case, so we *cannot* exploit maximal regularity theory to construct smooth solutions unless $\Gamma_2 \equiv \varnothing$). However, the proof can be based on the application of a Galerkin approximation scheme which is not standard due to the nature of the boundary condition (1.3). We refer the reader to, e.g., [7], for further details, where a system of reaction-diffusion equations for the phase-field equations with dynamic boundary conditions were considered (cf. also [26] in a degenerate case).

We begin the proof of Theorem 3.3 with a result for the eigenvalue problem for so-called Wentzell Laplacian Δ_W (see [22, Appendix]). More precisely, let us consider the equation

$$-\Delta\varphi = \Lambda\varphi \text{ in } \Omega, \tag{4.55}$$

with a boundary condition that depends on the eigenvalue Λ explicitly,

$$\partial_{\mathbf{n}}\varphi = \Lambda\varphi \text{ on } \Gamma_1, \tag{4.56}$$

such that

$$\partial_{\mathbf{n}}\varphi + \varphi = 0 \text{ on } \Gamma_2. \tag{4.57}$$

(recall that Γ_2 is assumed to be a set of positive measure and that (4.57) holds where a Dirichlet boundary condition for $u = u_i$ is satisfied on $\Gamma_2 \times \mathbb{R}_+$). Such a function φ will be called an eigenfunction associated with Λ and the set of all eigenvalues Λ of (4.55)-(4.57) will be denoted by Λ_j, $j \in \mathbb{N}$. Let $\varphi_1 \in C^2\left(\overline{\Omega}\right)$ and Λ_1, denote the principal eigenfunction and eigenvalue of (4.55)-(4.57), respectively. We have the following.

Proposition 4.1. *For the spectral problem (4.55)-(4.57),* $\Lambda_1 > 0$ *is simple and* $\varphi_1 > 0$ *in* $\overline{\Omega}$.

Proof. Using the standard characterization for the eigenvalues Λ_j of Δ_W (see, e.g., [22]), we obtain that the following min-max principle holds:

$$\Lambda_j = \min_{\substack{Y_j \subset H^1(\Omega), \\ \dim Y_j = j}} \max_{0 \neq \varphi \in Y_j} R_W\left(\varphi, \varphi\right), \ j \in \mathbb{N}, \tag{4.58}$$

where the Rayleigh quotient R_W, for the (boundary perturbed) Wentzell operators Δ_W, is given by

$$R_W\left(\varphi, \varphi\right) := \frac{\left\|\nabla\varphi\right\|_2^2 + \left\langle \varphi, \varphi \right\rangle_{L^2(\Gamma_2)}}{\left\|\varphi\right\|_{\mathbb{X}^2}^2}, \ 0 \neq \varphi \in H^1\left(\Omega\right). \tag{4.59}$$

Exploiting a well-known Friedrichs-Poincare's inequality, we have $R_W\left(\varphi, \varphi\right) \geq c_W \left\|\varphi\right\|_{\mathbb{X}^2}^2$, for some positive constant c_W, which implies that $\Lambda_j > 0$, for any $j \in \mathbb{N}$. By the maximum principle, φ_1 is positive in $\overline{\Omega}$ since Γ_2 has positive surface measure. The fact that Λ_1 is simple, follows again from the maximum principle (see, e.g., [4]). □

We are now ready to give the proof of Theorem 3.3. Without loss of generality, we can take $\delta_i = 1$, for all $i \in J_m$. We multiply (1.1) by $\left|u_i\right|^{m_i - 1} sgn\left(u_i\right) \varphi_1$, and

integrate over Ω, for each $i\in\mathbb{N}_m$. We obtain

$$\begin{aligned}(4.60)\qquad & \frac{1}{m_i}\frac{d}{dt}\int_\Omega |u_i|^{m_i}\varphi_1 dx+\left\langle f_i\left(x,t,\overrightarrow{u}\right),|u_i|^{m_i-1}sgn\left(u_i\right)\varphi_1\right\rangle_{L^2(\Omega)}\\ & -\int_\Omega div\left(a_i\left(u_i\right)\nabla u_i\right)|u_i|^{m_i-1}sgn\left(u_i\right)\varphi_1 dx\\ = & \ 0.\end{aligned}$$

Similarly, we multiply (1.3) by $|u_i|^{m_i-1}sgn\left(u_i\right)\varphi_1$ and integrate the relation over Γ (recall that (3.15) holds over Γ_2). We have

$$\begin{aligned}(4.61)\qquad & \frac{1}{m_i}\frac{d}{dt}\int_{\Gamma_1}|u_i|^{m_i}\varphi_1 dS+\int_\Gamma a_i\left(u_i\right)\partial_{\mathbf{n}}u_i|u_i|^{m_i-1}sgn\left(u_i\right)\varphi_1 dS\\ & +\left\langle g_i\left(x,t,\overrightarrow{u}\right),|u_i|^{m_i-1}sgn\left(u_i\right)\varphi_1\right\rangle_{L^2(\Gamma_1)}\\ & =0,\end{aligned}$$

for each $i\in\mathbb{N}_m$. Consider the following real-valued function

$$(4.62)\qquad \mathcal{E}\left(\overrightarrow{u}\left(x,t\right)\right)=\sum\nolimits_{i\in\mathbb{N}_m}\frac{1}{m_i}|u_i\left(x,t\right)|^{m_i}.$$

Integrating by parts in (4.60), then using (4.61), on account of the following computation

$$\begin{aligned}&\int_\Omega div\left(a_i\left(u_i\right)\nabla u_i\right)|u_i|^{m_i-1}sgn\left(u_i\right)\varphi_1 dx\\ &=-\left(m_i-1\right)\int_\Omega a_i\left(u_i\right)|\nabla u_i|^2|u_i|^{m_i-2}\varphi_1 dx\\ &-\int_\Omega a_i\left(u_i\right)|u_i|^{m_i-1}sgn\left(u_i\right)\nabla u_i\cdot\nabla\varphi_1 dx\\ &+\int_\Gamma a_i\left(u_i\right)\partial_{\mathbf{n}}u_i|u_i|^{m_i-1}sgn\left(u_i\right)\varphi_1 dS,\end{aligned}$$

we deduce the following inequality

$$\begin{aligned}(4.63)\qquad & \partial_t\int_{\overline{\Omega}}\mathcal{E}\left(\overrightarrow{u}\left(t\right)\right)\varphi_1 d\mu+\sum\nolimits_{i\in\mathbb{N}_m}\left(m_i-1\right)\int_\Omega a_i\left(u_i\right)|\nabla u_i|^2|u_i|^{m_i-2}\varphi_1 dx\\ & +\sum\nolimits_{i\in\mathbb{N}_m}\int_\Omega a_i\left(u_i\right)|u_i|^{m_i-1}sgn\left(u_i\right)\nabla u_i\cdot\nabla\varphi_1 dx\\ \leq & \ c\int_{\overline{\Omega}}\mathcal{E}\left(\overrightarrow{u}\left(t\right)\right)\varphi_1 d\mu+c.\end{aligned}$$

Here we have employed (3.16) to estimate the terms involving f_i, g_i in (4.60)-(4.61). Let us now estimate the third integral term on the left-hand side of (4.63). Exploiting the assumption (3.5) on a_i, $i\in\mathbb{N}_m$, we deduce for $u=u_i$, $p=p_i$, $M=m_i$ that

$$\begin{aligned}(4.64)\qquad \int_\Omega a\left(u\right)|u|^{p-1}sgn\left(u\right)\nabla u\cdot\nabla\varphi_1 dx &\geq\int_\Omega |u|^{p+M-1}sgn\left(u\right)\nabla u\cdot\nabla\varphi_1 dx\\ &=\int_\Omega\nabla\overline{a}\left(u\right)\cdot\nabla\varphi_1 dx,\end{aligned}$$

where we have set

$$\overline{a}\left(u\right):=\int_0^{|u|}a\left(s\right)|s|^{p-1}ds\geq c|u|^{M+p}.$$

Integrating by parts in (4.64) once more and noting that $\overline{a}(0)=0$, we obtain

$$\begin{aligned}\int_{\Omega}\nabla\overline{a}(u)\cdot\nabla\varphi_1dx &= \int_{\Gamma}\overline{a}(u)\,\partial_{\mathbf{n}}\varphi_1dS-\int_{\Omega}\overline{a}(u)\,\Delta\varphi_1dx\\ &= \int_{\Gamma_1}\overline{a}(u)\,\partial_{\mathbf{n}}\varphi_1dS+\int_{\Gamma_2}\overline{a}(u)\,\partial_{\mathbf{n}}\varphi_1dS\\ &\quad-\int_{\Omega}\overline{a}(u)\,\Delta\varphi_1dx\\ &= \int_{\Gamma_1}\overline{a}(u)\,\Lambda_1\varphi_1dS+\int_{\Omega}\overline{a}(u)\,\Lambda_1\varphi_1dx\\ &\geq \Lambda_1\int_{\overline{\Omega}}|u|^{M+p}\,\varphi_1d\mu,\end{aligned}\tag{4.65}$$

since (Λ_1,φ_1) satisfies (4.55)-(4.57). Inserting the above estimates in (4.63), we get the following inequality

$$\begin{aligned}&\partial_t\int_{\overline{\Omega}}\mathcal{E}\left(\overrightarrow{u}(t)\right)\varphi_1d\mu+\Lambda_1\sum\nolimits_{i\in\mathbb{N}_m}\int_{\overline{\Omega}}|u_i|^{m_i+p_i}\,\varphi_1d\mu\\ &\leq c\int_{\overline{\Omega}}\mathcal{E}\left(\overrightarrow{u}(t)\right)\varphi_1d\mu+c.\end{aligned}\tag{4.66}$$

Since all p_i's are positive, we can absorb the integral term on the right-hand side of (4.66), using the Young's inequality as follows:

$$\sum\nolimits_{i\in\mathbb{N}_m}|u_i|^{m_i}\leq\varepsilon\sum\nolimits_{i\in\mathbb{N}_m}|u_i|^{m_i+p_i}+C_{\varepsilon},$$

for a sufficiently small $\varepsilon\in(0,\Lambda_1)$ and some positive constant C_{ε}, independent of u_i,t. Moreover, setting $\nu=\min_{i\in\mathbb{N}_m}(m_i/p_i)+1>1$, we immediatelly have from the above inequality, that

$$\int_{\overline{\Omega}}\left(\mathcal{E}\left(\overrightarrow{u}(t)\right)\right)^{\nu}\varphi_1d\mu\leq c\sum\nolimits_{i\in\mathbb{N}_m}\int_{\overline{\Omega}}|u_i|^{m_i+p_i}\,\varphi_1d\mu+c,\tag{4.67}$$

for some positive constant c independent of $\overrightarrow{u}$, t and initial data. Using (4.67), we see that (4.66) yields the following estimate

$$\partial_t\int_{\overline{\Omega}}\mathcal{E}\left(\overrightarrow{u}(t)\right)\varphi_1d\mu+c\int_{\overline{\Omega}}\left(\mathcal{E}\left(\overrightarrow{u}(t)\right)\right)^{\nu}\varphi_1d\mu\leq c,\tag{4.68}$$

by an appropriate choice of $\varepsilon\leq\Lambda_1/2$. By normalizing the eigenfunction φ_1 in (4.68) such that $\|\varphi_1\|_{L^1(\overline{\Omega},d\mu)}=1$, on account of Jensen's inequality, it follows that

$$\left(\int_{\overline{\Omega}}\mathcal{E}\left(\overrightarrow{u}(t)\right)\varphi_1d\mu\right)^{\nu}\leq\int_{\overline{\Omega}}\left(\mathcal{E}\left(\overrightarrow{u}(t)\right)\right)^{\nu}\varphi_1d\mu,$$

which gives the following estimate:

$$\partial_tY(t)+c_1\left(Y(t)\right)^{\nu}\leq c_2,\tag{4.69}$$

for some positive constants c_1,c_2, where we have set

$$Y(t):=\int_{\overline{\Omega}}\mathcal{E}\left(\overrightarrow{u}(t)\right)\varphi_1d\mu.$$

We can now use the Gronwall's inequality (see, e.g., [37, Chapter III, Lemma 5.1]), applied to (4.69) to deduce that

$$Y(t)\leq\left(\frac{c_2}{c_1}\right)^{\nu}+\left(c_1(\nu-1)\,t\right)^{-\frac{1}{\nu-1}},\ \forall t>0,\tag{4.70}$$

which yields the desired claim. The proof of the theorem is complete.

Remark 4.2. In the case when $\overrightarrow{u}_0 \in \mathcal{X}^{\overrightarrow{r}}$, $Y(0) = \lim_{t\to 0^+} Y(t)$ is finite, so a similar argument to [37, Chapter III, Lemma 5.1] gives

$$Y(t) \leq \max\left\{ Y(0), \left(\frac{c_2}{c_1}\right)^{\nu} \right\}, \ \forall t \geq 0.$$

Thus, the second assertion in Theorem 3.3 also follows. We also note that the above argument relies entirely on the fact that the boundary Γ_2 has positive measure and this gives $\Lambda_1 > 0$. The proof fails to work if for instance, $\Gamma \equiv \Gamma_1$ (i.e., when $\Gamma_2 = \varnothing$). We shall require different arguments for this case (see below). Finally, if at least one $p_i = 0$, for some $i \in \mathbb{N}_m$, the above argument can still be used to derive the following bound

$$\sup_{t\geq 0} \|\overrightarrow{u}(t)\|_{\mathcal{X}^{\overrightarrow{r}}} \leq Q\left(\|\overrightarrow{u}_0\|_{\mathcal{X}^{\overrightarrow{r}}}\, e^{ct}\right).$$

Indeed, this follows from a standard application of Gronwall's inequality to (4.66).

We now continue with the proof of Theorem 3.4. As in the proof of Theorem 3.3, we multiply (1.1) by $|u_i|^{m_i-1} sgn(u_i)$, and integrate over Ω, for each $i \in \mathbb{N}_m$. Then, we multiply both equations (1.3) and (1.2) by $|u_i|^{m_i-1} sgn(u_i)$ and integrate the relations that we obtain over Γ. Analogously to (4.60)-(4.61) and arguing in a standard way as in (4.7), we deduce the following identity:

$$\begin{aligned}
(4.71)\qquad & \frac{d}{dt}\sum\nolimits_{i\in\mathbb{N}_m} \frac{1}{m_i}\left(\int_\Omega |u_i|^{m_i} dx + \int_\Gamma |u_i|^{m_i} dS\right) \\
&+ \sum\nolimits_{i\in\mathbb{N}_m} \left\langle f_i(x,t,\overrightarrow{u}), |u_i|^{m_i-1} sgn(u_i)\right\rangle_{L^2(\Omega)} \\
&+ \sum\nolimits_{i\in\mathbb{N}_m} \left\langle g_i(x,t,\overrightarrow{y}), |u_i|^{m_i-1} sgn(u_i)\right\rangle_{L^2(\Gamma)} \\
&= -\sum\nolimits_{i\in\mathbb{N}_m} (m_i-1)\int_\Omega a_i(x,t,\overrightarrow{u}) |\nabla u_i|^2 |u_i|^{m_i-2} dx.
\end{aligned}$$

By assumption (3.5), we can estimate the term on the right-hand side of (4.71) as follows:

$$\begin{aligned}
(4.72)\qquad \int_\Omega a_i(t,\overrightarrow{u}) |\nabla u_i|^2 |u_i|^{m_i-2} dx &\geq \alpha_i \int_\Omega |\nabla u_i|^2 |u_i|^{m_i+p_i-2} dx \\
&= \alpha_i \left(\frac{2}{m_i+p_i}\right)^2 \int_\Omega \left|\nabla |u|^{\frac{m_i+p_i}{2}}\right|^2 dx.
\end{aligned}$$

To estimate the nonlinear terms on the left-hand side of (4.71), we may exploit (3.17). On account of (4.72), we have

$$\begin{aligned}
(4.73)\qquad & \frac{d}{dt}\sum\nolimits_{i\in\mathbb{N}_m} \frac{1}{m_i}\left(\int_\Omega |u_i|^{m_i} dx + \int_\Gamma |u_i|^{m_i} dS\right) \\
&+ \sum\nolimits_{i\in\mathbb{N}_m} \alpha_i \left(\frac{2}{m_i+p_i}\right)^2 (m_i-1)\int_\Omega \left|\nabla |u|^{\frac{m_i+p_i}{2}}\right|^2 dx \\
&- \sum\nolimits_{i\in\mathbb{N}_m} \left(C_{f_i}\int_\Omega |u_i|^{m_i+p_i} dx + C_{g_i}\int_\Gamma |u_i|^{m_i+p_i} dS\right) \\
&\leq c.
\end{aligned}$$

By assumption (3.21), it follows that, for all $i \in \mathbb{N}_m$, it holds

$$a_i \|\nabla \varphi_i\|^2_{L^2(\Omega)} - C_{f_i} \|\varphi_i\|^2_{L^2(\Omega)} - C_{g_i} \|\varphi_i\|^2_{L^2(\Gamma)} \geq \Lambda_{1,i} \left(\|\varphi_i\|^2_{L^2(\Omega)} + \|\varphi_i\|^2_{L^2(\Gamma)} \right), \tag{4.74}$$

for all $\varphi_i \in H^1(\Omega)$. Thus, by choosing $\varphi_i = |u_i|^{(m_i+p_i)/2}$ in (4.74), and recalling (4.62), from (4.73), we obtain the following inequality:

$$\partial_t \int_{\overline{\Omega}} \mathcal{E}(\overrightarrow{u}(t))\, d\mu + \Lambda_1 \sum\nolimits_{i \in \mathbb{N}_m} \left(\int_\Omega |u_i|^{m_i+p_i} dx + C_{g_i} \int_\Gamma |u_i|^{m_i+p_i} dS \right) \leq c. \tag{4.75}$$

Let us assume that $p_i > 0$, for all $i \in \mathbb{N}_m$. Arguing now as in the proof of Theorem 3.3 (see (4.66)-(4.69)), it is not hard to see that we arrive at the following inequality

$$\partial_t \int_{\overline{\Omega}} \mathcal{E}(\overrightarrow{u}(t))\, d\mu + c \left(\int_{\overline{\Omega}} \mathcal{E}(\overrightarrow{u}(t))\, d\mu \right)^v \leq c, \tag{4.76}$$

where $v = \min_{i \in \mathbb{N}_m} (m_i/p_i) + 1 > 1$. Thus, we can use the Gronwall's inequality as before (see (4.70)) to derive the estimate

$$\int_{\overline{\Omega}} \mathcal{E}(\overrightarrow{u}(t))\, d\mu \leq c \left(1 + t^{-\frac{1}{\nu-1}}\right), \ \forall t > 0. \tag{4.77}$$

Hence, the first claim of the theorem follows from (4.77). On the other hand, if at least one $p_i = 0$ for some $i \in \mathbb{N}_m$, we obtain the following analogue of (4.76):

$$\partial_t \int_{\overline{\Omega}} \mathcal{E}(\overrightarrow{u}(t))\, d\mu + c \int_{\overline{\Omega}} \mathcal{E}(\overrightarrow{u}(t))\, d\mu \leq c, \tag{4.78}$$

which yields the second claim of the theorem once more on account of Gronwall's inequality. In particular, there exists a positive function Q, independent of initial data and time, such that

$$\sup_{t \geq 0} \|\overrightarrow{u}(t)\|_{\mathcal{X}^{\overrightarrow{r}}} \leq Q(\|\overrightarrow{u}_0\|_{\mathcal{X}^{\overrightarrow{r}}})\, e^{-c_0 t} + C_0, \tag{4.79}$$

for some positive constants c_0, C_0 independent of initial data and time. The proof of Theorem 3.4 is now complete.

5. Appendix

We will consider a more general problem than (1.7)-(1.8) by taking $f_1(s) = f(s) - \lambda s$, $g_1(s) = g(s) - \gamma s$, provided that $\lambda, \gamma > 0$ are sufficiently large, and

$$f(s) \geq -c_f,\ g(s) \geq -c_g,\ \forall s \in \mathbb{R},$$

for some positive constants c_f, c_g. If $f(s) \sim C_f |s|^p s$ and $g(s) \sim C_g |s|^q s$, as $|s| \to \infty$, for $p, q > 1$ and some positive constants C_f, C_g, it is well-known [23] that problem

$$\partial_t u - \nu \Delta u + f(u) - \lambda u = 0, \text{ in } \Omega \times (0, +\infty), \tag{5.1}$$

subject to the dynamic condition

$$\partial_t u + \nu \partial_{\mathbf{n}} u + g(u) - \gamma u = 0, \text{ on } \Gamma \times (0, \infty), \tag{5.2}$$

and initial condition

$$u_{|t=0} = u_0 \text{ in } \overline{\Omega}, \tag{5.3}$$

possesses a finite dimensional global attractor $\mathcal{A}_{\rm dyn}$ which is bounded in $H^2(\Omega) \cap \mathbb{X}^{\infty}$.

Let u_* be a constant (hyperbolic) equilbrium for the system (5.1)-(5.2) (see [22, Section 3]). We linearize (5.1)-(5.2) around u_*. We obtain

$$\partial_t u = \nu \Delta u - \left(f'(u_*) - \lambda\right) u, \text{ in } \Omega \times (0, +\infty), \tag{5.4}$$

subject to the dynamic condition

$$\partial_t u = -\nu \partial_{\mathbf{n}} u - \left(g'(u_*) - \gamma\right) u, \text{ on } \Gamma \times (0, \infty). \tag{5.5}$$

We aim to better understand the nature of the (invariant) unstable eigenspace E^u which corresponds to the following (matrix) operator

$$\mathbf{L}(u_*) W = \begin{pmatrix} \nu \Delta w - f'(u_*) w + \lambda w \\ -\nu \partial_{\mathbf{n}} w - g'(u_*) w + \gamma w \end{pmatrix}, \; W = \begin{pmatrix} w \\ w_{|\Gamma} \end{pmatrix},$$

with $\sigma(\mathbf{L}(u_*)) \subset \{\zeta : \zeta > 0\}$. We note that $(\mathbf{L}(u_*), \mathrm{dom}(\mathbf{L}(u_*)))$ is self-adjoint on $\mathbb{X}^2$ with spectrum contained in $(-\infty, C_{\lambda,\gamma}]$, for some $C_{\lambda,\gamma} > 0$ which depends only on f, g, λ and γ (see, e.g., [22] and references therein). Next, let $\left\{\varphi_j(x)\right\}_{j \in \mathbb{N}_0}$ be an orthonormal basis in $\mathbb{X}^2$ consisting of eigenfunctions of the (positive) Wentzell Laplacian Δ_W operator (see [22, Theorem 5.1])

$$\Delta_W \varphi_j = \Lambda_j \varphi_j, \; j \in \mathbb{N}_0, \; \varphi_j \in \mathrm{dom}(\Delta_W) \cap C\left(\overline{\Omega}\right) \tag{5.6}$$

such that

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq ... \leq \Lambda_{,j} \leq \Lambda_{j+1} \leq .... \to +\infty.$$

We shall seek for eigenvectors $W_j = \binom{w_j}{w_{j|\Gamma}} \in \mathbb{X}^2$, of the form $w_j(x) = \varphi_j(x) p_j$, $p_j \in \mathbb{R}$, satisfying equation

$$\mathbf{L}(u_*) W_j = \zeta_j W_j, \; W_j \in \mathrm{dom}(\mathbf{L}(u_*)) := \mathrm{dom}(\Delta_W). \tag{5.7}$$

Note that for $W_j \in \mathit{dom}(\mathbf{L}(u_*)) \subset H^1(\Omega) \times L^2(\Gamma)$, the trace of w_j makes sense as an element of $H^{1/2}(\Gamma)$. Substituting such w_j into (5.7), taking into account (5.6) and the fact that

$$\mathbf{L}(u_*) W_j = -\nu \Delta_W W_j + \Pi_{\lambda,\gamma} W_j, \; \Pi_{\lambda,\gamma} W_j := \begin{pmatrix} (-f'(u_*) + \lambda) w_j \\ (-g'(u_*) + \gamma) w_{j|\Gamma} \end{pmatrix},$$

we obtain the equation

$$(-\nu \Lambda_j I + \Pi_{\lambda,\gamma}) p_j = \zeta_j p_j, \; \Pi_{\lambda,\gamma} = \begin{pmatrix} -f'(u_*) + \lambda & 0 \\ 0 & -g'(u_*) + \gamma \end{pmatrix}. \tag{5.8}$$

A nonzero p_j exists if $\zeta = \zeta_j$ is a root of the equation

$$\det(-\nu \Lambda_j I + \Pi_{\lambda,\gamma} - \zeta I) = 0, \; \zeta > 0. \tag{5.9}$$

When $\nu = 0$, this equation has at least one root $\zeta > 0$ provided that at least one of λ and γ is sufficiently large, i.e., either $\lambda > f'(u_*)$ or $\gamma > g'(u_*)$ (in fact the roots are $\zeta = \lambda - f'(u_*)$ and $\zeta = \gamma - g'(u_*)$, respectively). Therefore, there exists $\delta > 0$ such that when $\nu \Lambda_j < \delta$, the equation (5.9) has a root $\zeta_j(\mathbf{L}) = \zeta_j(\nu)$ with $\zeta_j > 0$. Therefore, to any such root ζ_j, we can assign a nontrivial p_j, which is a solution of (5.8), and thus an eigenvector W_j. Let us now compute how many j's satisfy the inequality $\nu \Lambda_j < \delta$. When $N \geq 3$, the asymptotic behavior of Λ_j is

$$\Lambda_j \sim C_S(\Gamma) j^{1/(N-1)} \text{ as } j \to \infty \tag{5.10}$$

(see [22, Theorem 5.4]). The inequality $\nu\Lambda_j < \delta$ certainly holds when

$$1 \leq j \leq C_{\lambda,\gamma}\delta^{n-1}\left(C_S\left(\Gamma\right)\nu\right)^{1-N} = C_{\lambda,\gamma}^{'}\left|\Gamma\right|\left(\frac{1}{\nu}\right)^{N-1}, \text{ for } N \geq 3, \tag{5.11}$$

where the positive constants $C_{\lambda,\gamma}, C_{\lambda,\gamma}^{'}$ depend only on λ, γ and N.

Remark 5.1. Note that the number of unstable mode solutions to (5.4)-(5.5) obeys the same relation (5.11) even when $f \equiv 0$ and $\lambda = 0$ in (5.1) (i.e., the dynamics of u inside the bulk Ω is strictly linear). Finally, we note that both $C_{\lambda,\gamma}, C_{\lambda,\gamma}^{'} \to +\infty$ if either $\gamma \to +\infty$ or $\lambda \to +\infty$ (cf. also [22, Section 3]). In this case the instability index of u_* is

$$N_+\left(u_*\right) \sim C_{\lambda,\gamma}^{'}\left|\Gamma\right|\left(\frac{1}{\nu}\right)^{N-1}, \; N \geq 3.$$

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